Vector and Metric Spaces: Inner Products, Norms, Metrics, and All That

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Abstract

This note goes over inner products and norms defined on vector spaces and how these may relate to metric spaces. Hopefully, some of the confusion in the minds of mathematics and engineering students will be resolved.

"Every big idea needs someone to defend it."
— Cybersecurity Company

1 Introduction

This note will present three main definitions: Inner product space, a normed vector space, and what a metric space is, and finally, how to go from one of these to one of the others, when it is possible to do so. I won't here define what a vector space is, but rather assume that the reader already knows what that is. Further, since the purpose of this note is to demonstrate the connections between metric spaces, and normed vectors spaces and inner product spaces, I will simplify the description by restricting the field of the scalars of the vector spaces to the real numbers $\mathbb{R}$. So, be advised that there are some subtle changes that need to be made when the field of the vector space is complex.

In pure mathematics, a shiny new vector space, right out of the box, knows nothing about the length of vectors or angles between them. Typically, to an engineer or a physicist, a vector space needs to be endowed by either a norm or an inner product. The norm is a mapping of every vector to the reals that provides a notion of vector length. An inner product provides a means to define lengths and angle between vectors.

What all three of these spaces have in common is some mapping on them that takes in one or more elements of the space and maps it to a real number (more generally to a complex number).

Definition: A *functional* on a space is a mapping of the points or vectors of the space to the real numbers.

2 A metric space

Definition: A *metric* on a set of points M is a means of providing a 'distance' between those points. So, on any two points x, y, z of this space M, we define the functional $d(x,y) \to \mathbb{R}$, such that

(i) $d(x,y) \geq 0 \;\forall\; x, y \;\in\; M\,,$

(ii) $d(x,y) = 0 \iff x = y \;\forall\; x, y \;\in\; M\,,$

(iii) $d(x,y) = d(y,x) \;\forall\; x, y \;\in\; M\,,$

(iv) $d(x,z) \leq d(x,y) + d(y,z) \;\forall\; x, y, z \;\in\; M\,.$

3 A normed vector space

Definition: A *norm* on a vector in a vector space $\mathcal{V}$ is a means of providing a 'length' to a vector. So, on any vectors u, v of $\mathcal{V}$, and for all $\alpha \in \mathbb{R}$ we define the functional $\|\cdot\| \to \mathbb{R}$, such that

(i) $\|\, v \,\| \geq 0\,,$

(ii) $\|\, v \,\| = 0$ iff $v = 0\,,$

(iii) $\|\, \alpha v \,\| = |\, \alpha \,|\, \|\, v \,\|$, where $|\, \alpha \,|$ is the absolute value function,

(iv) $\|\, v + u \,\| \leq \|\, v \,\| + \|\, u \,\|\,.$

Now, there are two ways to establish that a vector space $\mathcal{V}$ is a normed vector space. The first is to simply declare it so. The second is to declare that $\mathcal{V}$ is endowed with a certain property X, which can be used to show that a norm can be defined on $\mathcal{V}$. And in the latter case, we have to show that property X can fulfil all the required properties of a norm defined by the above list.

4 An inner product space

Definition: A *inner product* on a vector in a vector space $\mathcal{V}$ is a means of providing a 'length' to a vector and angles between vectors. So, on any vectors x, y, z of $\mathcal{V}$, and for all $\alpha, \beta \in \mathbb{R}$ we define the functional $\langle \cdot, \cdot \rangle \to \mathbb{R}$, such that

(i) $\langle\, x, x \,\rangle \geq 0\,,$

(ii) $\langle\, x, x \,\rangle = 0$ iff $x = 0\,,$

(iii) $\langle\, x, y \,\rangle = \langle\, y, x \,\rangle\,,$

(iv) $\langle\, \alpha x + \beta y, z \,\rangle = \alpha \langle\, x, z \,\rangle + \beta \langle\, y, z \,\rangle\,.$

As in the previous case, there are two ways to establish that a vector space $\mathcal{V}$ is an inner product space. The first is to simply declare it so, and the second is to demonstrate it.

Now, it's probably clear to the reader that the inner product is a generalization of the dot product of Gibbs's vector algebra, such as the form $\mathbf{a} \cdot \mathbf{b}$. The study of inner product spaces is a huge field of study that is central to both mathematics and mathematical physics.

5 The Game Plan

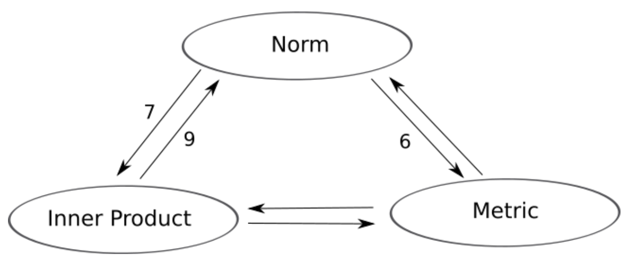

Figure 1. If we interpret these arrows as implications, how many of them are true? By which I mean, given one feature, can we deduce the other? The game plan for the rest of this paper is to find out. The numbers by the arrows indicate the section numbers where these are established.

The rest of this note is dedicated to answering questions such as: If we have a normed vector space, can we use that to make it an inner product space, or vice versa.

6 A normed space is a metric space

Let N be a normed vector space with norm $\|\cdot\|$. So, we have to be clever to think of a way to change a functional whose purpose is to give us the lengths of vectors into a functional whose purpose is to give us a distance between any two points of the vector space. We must first answer the question: What 'points' are we even talking about? Vector spaces have vectors, not points.

Let's assume that our vector space is n-dimensional. Then we attach to the end of each vector tip a point of an n-dimensional space. We can now define a vector between any two such points as the vector difference of the vectors which define each point. So, if one point is marked by vector x and the other by vector y, then we can define the presumptive metric

$$d(x, y) \equiv \| x - y \| , \tag{1}$$

where we now have to prove that (1) does indeed define a metric on N.

(i) $d(x,y) \geq 0 \ \forall\ x,y \in M$ because $\| x-y \| \geq 0 \ \forall\ x,y \in M$,

(ii) $d(x,y) = 0$ iff $x = y \ \forall\ x,y \in M$ because $\| x-y \| = 0$ iff $x = y \ \forall\ x,y \in M$,[1]

(iii) $d(x,y) = d(y,x) \ \forall\ x,y \in M$ because $\| x-y \| = \| y-x \| \ \forall\ x,y \in M$,

(iv) $d(x,z) \leq d(x,y) + d(y,z) \ \forall\ x,y,z \in M$ because
$\| x-z \| \leq \| x-y \| + \| y-z \| \ \forall\ x,y,z \in M$.

7 A normed space is an inner product space

Once more we're going to act presumptively, this time by assuming that we can define an inner product on a normed space by

$$\langle v,w \rangle \equiv \frac{\| v+w \|^2 - \| v-w \|^2}{4}, \tag{2}$$

and, of course, this has to be proven.[2]

Let's begin with the inner product of an arbitrary vector v with itself:

$$\langle v,v \rangle = \frac{\| 2v \|^2 - 0}{4} = \| v \|^2 . \tag{3}$$

That looks like a good start, but now we're going to need a couple identities out of the domain of inner products:

$$\langle v+w, v+w \rangle = \langle v,v \rangle^2 + 2\langle v,w \rangle + \langle w,w \rangle^2 , \tag{4}$$

$$\langle v-w, v-w \rangle = \langle v,v \rangle^2 - 2\langle v,w \rangle + \langle w,w \rangle^2 . \tag{5}$$

Now, using (3), we get

$$\| v+w \|^2 = \| v \|^2 + 2\langle v,w \rangle + \| w \|^2 , \tag{6}$$

$$\| v-w \|^2 = \| v \|^2 - 2\langle v,w \rangle + \| w \|^2 . \tag{7}$$

And we can establish (2) by solving these last two equations for $\langle v,w \rangle$.

Now for the details.

$$\langle v,v \rangle = \| v \|^2 \geq 0 , \tag{8}$$

hence,

$$\langle v,v \rangle = 0 \quad \text{iff} \quad v = 0 . \tag{9}$$

[1]The only vector whose norm is zero is the zero vector. Hence, $\| x-y \| = 0$ iff $x-y = 0$ iff $x = y$.

[2]This 'identity' has the name 'polarization identity', for reasons I can't figure out. This identity can be generalized to a *symmetric bilinear form*, which I won't go into in this paper.

And,

$$\begin{aligned}
\langle x, y \rangle &= \frac{\| x + y \|^2 - \| x - y \|^2}{4} \\
&= \frac{\| y + x \|^2 - \| y - x \|^2}{4} \\
&= \langle y, x \rangle \,. \qquad (10)
\end{aligned}$$

Finally, we need to show that

$$\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle \,, \qquad (11)$$

in terms of Eqs. (2)–(10). So,

$$\begin{aligned}
4\langle \alpha x + \beta y, z \rangle &= \| (\alpha x + \beta y) + z \|^2 - \| (\alpha x + \beta y) - z \|^2 \\
&= \| \alpha x + \beta y \|^2 + 2\langle \alpha x + \beta y, z \rangle + \| z \|^2 \\
&\quad - [\| \alpha x + \beta y \|^2 - 2\langle \alpha x + \beta y, z \rangle + \| z \|^2] \\
&= 4[\alpha \langle x, z \rangle + \beta \langle y, z \rangle] \,. \qquad (12)
\end{aligned}$$

Now, divide through by 4.

Although we've already made our proof, I'm going to add a corollary, called the *Parallelogram Law*:

$$\| v + w \|^2 + \| v - w \|^2 = 2(\| v \|^2 + \| w \|^2) \,. \qquad (13)$$

8 Lemma: Cauchy-Schwarz Inequality (CSI)

Given a real inner-product space V, with arbitrary vectors $u, v \in V$, then the Cauchy-Schwarz Inequality theorem claims that

$$\langle v, u \rangle^2 \le \langle v, v \rangle \langle u, u \rangle \qquad (14)$$

Now, the proof has one tricky element to it, but it is easy to follow. Let t be a possibly nonreal parameter, then, since our inner product is positive definite, let $p(t)$ be the real-valued polynomial

$$p(t) = \langle tu + v, tu + v \rangle \ge 0 \,. \qquad (15)$$

Expanding this, we have that

$$p(t) = t^2 \langle u, u \rangle + 2t \langle u, v \rangle + \langle v, v \rangle \ge 0 \,. \qquad (16)$$

If we introduce the temporary variables $a = \langle u, u \rangle$, $b = 2\langle u, v \rangle$, and $c = \langle v, v \rangle$, we can rewrite the last equation as

$$at^2 + bt + c = 0 \,. \qquad (17)$$

On solving for t, we get

$$t = \frac{-2\langle u, v \rangle \pm \sqrt{(2\langle u, v \rangle)^2 - 4\langle u, u \rangle\langle v, v \rangle}}{2\langle u, u \rangle} \tag{18}$$

Now, since $p(t)$ is never negative, it has no roots other than a possible double root (when the discriminant is zero). Thus, the discriminant is negative to ensure only complex roots (this guarantees that the graph of $p(t)$ does not cross the x-axis). Hence,

$$\langle u, v \rangle^2 - \langle u, u \rangle\langle v, v \rangle \leq 0\,. \tag{19}$$

On rearranging, we get

$$\langle u, v \rangle^2 \leq \langle u, u \rangle\langle v, v \rangle\,. \tag{20}$$

On taking the square root of both sides, we have that

$$\langle u, v \rangle \leq \sqrt{\langle u, u \rangle}\sqrt{\langle v, v \rangle}\,. \tag{21}$$

9 An inner product space is a normed space

Let V be an inner product space with typical vectors v, u. Let α be an arbitrary field element. We now presumptively define the norm of vector v by

$$\| v \| \equiv \sqrt{\langle v, v \rangle} \geq 0\,. \tag{22}$$

Is this definition consistent with all the properties of a norm? Let's work it out. First, $\| v \| = 0$ iff $v = 0$. Second,

$$\| \alpha v \| = \sqrt{\langle \alpha v, \alpha v \rangle} = \sqrt{\alpha^2\langle v, v \rangle} = |\alpha| \sqrt{\langle v, v \rangle} = |\alpha| \, \| v \|\,. \tag{23}$$

Third,

$$\begin{aligned} \| v + u \|^2 &= \langle v + u, v + u \rangle \\ &= \langle v, v \rangle^2 + 2\langle v, u \rangle + \langle u, u \rangle^2 \\ &= \| v \|^2 + 2\langle v, u \rangle + \| u \|^2 \\ &\leq \| v \|^2 + 2 \| v \| \, \| u \| + \| u \|^2 \quad \text{(by CSI)} \\ &= [\, \| v \| + \| u \| \,]^2 \end{aligned} \tag{24}$$

Now, we take the square root of both sides to get

$$\| v + u \| \leq \| v \| + \| u \|\,. \tag{25}$$

10 Every finite-dimensional inner-product space can be granted an orthonormal basis

We haven't yet established the meaning of the **orthogonality** of two vectors, say x and y. Vectors x and y are said to be orthogonal to each other when $\langle x, y \rangle = 0$.

But we also need to deal with the normality of vectors to create an orthonormal set. It's a simple matter to set up a procedure of vector normalization by introducing the 'square of a vector' (i.e., its inner product with itself) so that the resulting unit vector $\hat{x}$ 'squares' to unity. Since we can define a norm of a vector x in an inner product space, $\| x \|$, we can normalize a nonzero x by the procedure

$$\hat{x} \equiv \frac{x}{\| x \|} = \frac{x}{\sqrt{\langle x, x \rangle}} \quad \text{where} \quad x \neq 0\,. \tag{26}$$

Then

$$\hat{x}^2 \equiv \langle \hat{x}, \hat{x} \rangle = \langle \frac{x}{\| x \|}, \frac{x}{\| x \|} \rangle = \frac{\langle x, x \rangle}{\| x \|^2} = \frac{\langle x, x \rangle}{\langle x, x \rangle} = 1\,. \tag{27}$$

Okay, now that we know how to normalize vectors, can we make them perpendicular to each other? That is to say, can we exchange a non-orthogonal set with an orthogonal set?

We'll begin by examining a basis for a 2-D space, $\{a, b\}$. We're going to assume that these two vectors are unit vectors. If they aren't, we'll convert them to unit vectors before continuing. Now, let

$$A = a + b\,, \quad B = a - b\,. \tag{28}$$

Then

$$\begin{aligned} \langle A, B \rangle &= \langle a + b, a - b \rangle \\ &= \langle a, a \rangle + \langle a, (-b) \rangle + \langle b, a \rangle + \langle b, (-b) \rangle \\ &= \langle a, a \rangle - \langle a, b \rangle + \langle a, b \rangle - \langle b, b \rangle = 0 \\ &= 1 - 1 = 0\,. \end{aligned} \tag{29}$$

Thus, vectors A and B are mutually orthogonal. All we have to do now is to normalize them:

$$\begin{aligned} A \to A &= \frac{a + b}{\| a + b \|}\,, \\ B \to B &= \frac{a - b}{\| a - b \|}\,. \end{aligned} \tag{30}$$

Next, we look at a better way to orthogonalize a finite set of basis vectors, called the *Gram–Schmidt Process*.

Say that we are given a random set of n basis vectors to our n-dimensional vector space, $\{a_1, a_2, \ldots, a_n\}$. We wish to replace this set with a set of mutually orthogonal vectors, $\{A_1, A_2, \ldots, A_n\}$. There is a methodological way to do this.

The first thing we do is to take the first basis vector a_1 and normalize it to get A_1. We will normalize each new vector as we derive it.

$$A_1 = a / \| a_1 \|\,. \tag{31}$$

Now we have our first vector. To find the next vector A_2, take the next original basis vector a_2, and combine it with A_1 to get

$$A_2 = \alpha A_1 + \gamma a_2\,, \tag{32}$$

subject to the constraint that $\langle A_2, A_1 \rangle = 0$.[3] Therefore,

$$\langle A_2, A_1 \rangle = \alpha + \gamma \langle a_2, A_1 \rangle = 0 \,. \tag{33}$$

Solving for α, we get

$$\alpha = -\gamma \langle a_2, A_1 \rangle \,. \tag{34}$$

On substituting this into (32), we have that

$$A_2 = \gamma [\, a_2 - \langle a_2, A_1 \rangle A_1 \,] \,, \tag{35}$$

where we have to determine what we're going to do with γ. I will normalize these new basis vectors as we go. Thus, I will set $\gamma = \dfrac{1}{\| \% \|}$, where the percent symbol holds the place of the vector quantity to its right. The effect of this is that A_k will be normalized as we go, and thus $\langle A_k, A_k \rangle = 1$ for all $k \leq n$. Therefore,

$$A_2 = \frac{a_2 - \langle a_2, A_1 \rangle A_1}{\| a_2 - \langle a_2, A_1 \rangle A_1 \|} \,. \tag{36}$$

So, now we move to the next vector A_3. Let's express A_3 as a linear combination of A_1, A_2, and a_3:

$$A_3 = \alpha A_1 + \beta A_2 + \gamma a_3 \,, \tag{37}$$

with necessary constraints on α, β, γ, given by

$$\langle A_3, A_1 \rangle = 0 \quad \text{and} \quad \langle A_3, A_2 \rangle = 0 \,. \tag{38}$$

On applying the operator $\langle \cdot, A_1 \rangle$ on both sides of (37), we get

$$0 = \langle A_3, A_1 \rangle = \alpha \langle A_1, A_1 \rangle + \gamma \langle a_3, A_1 \rangle \,, \tag{39}$$

from which we get

$$\alpha = -\gamma \langle a_3, A_1 \rangle / \langle A_1, A_1 \rangle = -\gamma \langle a_3, A_1 \rangle \,. \tag{40}$$

By a similar process we get

$$\beta = -\gamma \langle a_3, A_2 \rangle / \langle A_2, A_2 \rangle = -\gamma \langle a_3, A_2 \rangle \,. \tag{41}$$

On substituting these values into (37) we have that

$$A_3 = \gamma \Big[\, a_3 - \langle a_3, A_1 \rangle A_1 - \langle a_3, A_2 \rangle A_2 \,\Big] . \tag{42}$$

And on setting γ equal to the inverse of the norm of the vector to its right, we get

$$A_3 = \frac{a_3 - \langle a_3, A_1 \rangle A_1 - \langle a_3, A_2 \rangle A_2}{\| a_3 - \langle a_3, A_1 \rangle A_1 - \langle a_3, A_2 \rangle A_2 \|} \,. \tag{43}$$

[3] Usually, demonstrations of the Gram–Schmidt Process include diagrams of the projection of this vector onto that vector, which is fine. But the proof itself is purely algebraic.

We can imagine extending this procedure so that if we have calculated the first $k-1$ orthonormal vectors from the original set, that we can then calculate the kth element of the new set (i.e., A_k), by the formula

$$A_k = \frac{a_k - \sum_{i=1}^{k-1} \langle\, a_i, A_i \,\rangle A_i}{\left\| a_k - \sum\limits_{i=1}^{k-1} \langle\, a_i, A_i \,\rangle A_i \right\|}, \tag{44}$$

which satisfies the constraints

$$\langle\, A_i, A_j \,\rangle = \delta_{ij} \quad \forall\; i,j \in \{1,2,\ldots,k-1\}\,. \tag{45}$$

Finally, once we have A_n, we're finished collecting the new set of orthonormal basis vectors.

11 A practice problem for the Gram–Schmidt Process

Now, I'll perform a practice problem, taken from one of my original linear algebra textbooks *Elementary Linear Algebra*, 2nd Ed., Example Problem 58, pg. 179–180.[4]

> Consider the vector space R^3 with Euclidean inner product. Apply the Gram–Schmidt process to transform the basis $\mathbf{a}_1 = (1,1,1)$, $\mathbf{a}_2 = (0,1,1)$, $\mathbf{a}_3 = (0,0,1)$ into an orthonormal basis. (I relabeled the basis vectors according to my own version of the Gram–Schmidt formula.)

Step 1: Take vector 1 of the original basis and normalize it.

$$\|\, \mathbf{a}_1 \,\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}\,, \tag{46}$$

which yields

$$\mathbf{A}_1 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}. \tag{47}$$

Step 2: Take vector 2 of the original basis and perform the following operation

[4] *Elementary Linear Algebra*, 2nd Ed., Howard Anton, Wiley & Sons, 1977. (This book was very new when I took it in college.)

to get $\mathbf{A}_2$:

$$\begin{aligned}\mathbf{A}_2 &= \gamma[\, \mathbf{a}_2 - \langle\, \mathbf{a}_2, \mathbf{A}_1 \,\rangle \mathbf{A}_1 \,] \\ &= \gamma\Big[\begin{pmatrix}0\\1\\1\end{pmatrix} - (0,1,1)\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix} \Big] \\ &= \gamma\Big[\begin{pmatrix}0\\1\\1\end{pmatrix} - \frac{2}{\sqrt{3}}\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix} \Big] = \gamma\begin{pmatrix}-2/3\\1/3\\1/3\end{pmatrix}. \end{aligned} \tag{48}$$

Now,

$$\gamma = 1/\sqrt{(-2/3)^2 + (1/3)^2 + (1/3)^2} = \sqrt{\frac{3}{2}}\,. \tag{49}$$

Hence,

$$\mathbf{A}_2 = \sqrt{\frac{3}{2}}\begin{pmatrix}-2/3\\1/3\\1/3\end{pmatrix} = \begin{pmatrix}-2/\sqrt{6}\\1/\sqrt{6}\\1/\sqrt{6}\end{pmatrix}. \tag{50}$$

Step 3: Take vector 3 of the original basis and perform the following operation to get $\mathbf{A}_3$:

$$\begin{aligned}\mathbf{A}_3 &= \gamma[\, \mathbf{a}_3 - \langle\, \mathbf{a}_3, \mathbf{A}_1 \,\rangle \mathbf{A}_1 - \langle\, \mathbf{a}_3, \mathbf{A}_2 \,\rangle \mathbf{A}_2 \,] \\ &= \gamma\Big[\begin{pmatrix}0\\0\\1\end{pmatrix} - (0,0,1)\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix} - (0,0,1)\begin{pmatrix}-2/\sqrt{6}\\1/\sqrt{6}\\1/\sqrt{6}\end{pmatrix}\begin{pmatrix}-2/\sqrt{6}\\1/\sqrt{6}\\1/\sqrt{6}\end{pmatrix} \Big] \\ &= \gamma\Big[\begin{pmatrix}0\\0\\1\end{pmatrix} - \begin{pmatrix}1/3\\1/3\\1/3\end{pmatrix} - \begin{pmatrix}-1/3\\1/6\\1/6\end{pmatrix} \Big] \\ &= \gamma\begin{pmatrix}0\\-1/2\\1/2\end{pmatrix}. \end{aligned} \tag{51}$$

Now,

$$\gamma = 1/\sqrt{(0)^2 + (-1/2)^2 + (1/2)^2} = \frac{2}{\sqrt{2}}\,. \tag{52}$$

And, so, the result for $\mathbf{A}_3$ is

$$\mathbf{A}_3 = \begin{pmatrix}0\\-1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}. \tag{53}$$

Done.

12 A bit more on normed vector spaces

The reader may be most familiar with the Euclidean norm of a vector v, in an inner product space of dimension n, with orthonormal basis $\{e_1, e_2, \ldots, e_n\}$, where

$$v = \sum_{i=1}^{n} v_i e_i \,. \tag{54}$$

(So, now the indices are labeling vector components, not distinct vectors.) Thus, the norm is given as

$$\| \, v \, \| \equiv \sqrt{\sum_{i=1}^{n} v_i^2} = [\, \sum_{i=1}^{n} v_i^2 \,]^{1/2} \,. \tag{55}$$

The idea of this being that we take first the sum of squares of the components of a vector and then take the square root of that. This norm is referred to as an L^2 (or ℓ^2) norm. Generalizing, we can define an L^p (or ℓ^p) norm by[5]

$$\| \, v \, \|_p \equiv \Big[\, \sum_{i=1}^{n} | \, v_i \, |^p \, \Big]^{1/p} \,, \tag{56}$$

where p is any real number ≥ 1.

Now, there are many useful notions of norms on vector spaces, and one is often required to subscript them to differentiate them, especially if one is forced to mix together different norms in a given problem. The proof that ℓ^p is actually a norm is quite involved and won't be given here.

Let's expand (56), to get

$$\| \, v \, \|_p = \Big[\, | \, v_1 \, |^p + | \, v_2 \, |^p + \cdots + | \, v_n \, |^p \, \Big]^{1/p} \,. \tag{57}$$

Now let's consider the limit of $\| \, v \, \|_p$ as $p \to \infty$ (the ℓ-infinity norm):

$$\| \, v \, \|_\infty \equiv \lim_{p \to \infty} \Big[\, | \, v_1 \, |^p + | \, v_2 \, |^p + \cdots + | \, v_n \, |^p \, \Big]^{1/p} \,. \tag{58}$$

Think about what happens to the sum of terms inside the square brackets prior to taking the pth root. Effectively, the largest term of the sum will dominate so completely that the rest of the terms can be ignored, in which case one gets

$$\| \, v \, \|_\infty = \max\{ | \, v_1 \, |, | \, v_2 \, |, \ldots, | \, v_n \, | \} \,. \tag{59}$$

The last norm I want to address at this time is the ℓ^1 norm, given by

$$\| \, v \, \|_1 = | \, v_1 \, | + | \, v_2 \, | + \cdots + | \, v_n \, | \,. \tag{60}$$

[5] Some reserve the capital L for norms on function spaces, while using the small ℓ for norms on vectors.

13 Conclusion

The definitions for normal vector space, inner-product space, and metric space are so similar, learning about them together can be confusing. My goal in this paper was first to make their distinctiveness apparent among them and then provide a few important theorems.