An Instructive Unipodal Integral

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1 Getting Started

Our job is to perform the integral

$$I = \int (a \cosh x - \sinh x) \cosh bx \, dx \,. \tag{1}$$

Now, no doubt one could just break up this integral into two integrals, each simpler than the original, but I think it's instructive to use the unipodal algebra to do the integral without breaking it up. Just as one can obtain aid to integrate trigonometric functions by converting them to complex exponentials, so one can obtain aid to integrate hyperbolic functions by replacing them by unipodal exponentials, that is, exponentials that have a unipotent element in their exponent.

The next section provides an introduction to the unipodal algebra. However, if you are already familiar with the algebra, you can skip down to Section 3.

2 Getting to know the Unipodal Algebra

The unipodal algebra is formed by all linear combinations of the two basis vectors $\{1, u\}$ (forming the *standard basis*), where u is some number not ± 1 whose square is unity, making it a unipotent number.

$$u^2 = 1 \,. \tag{2}$$

There is an alternative basis, called the *idempotent basis*, which is given by $\{u_+, u_-\}$, which will be defined below.

Every unipodal number X, or *unipode*, in standard form can be expressed in the form $X = a + bu$, where a and b are complex numbers, respectively called the complex and uniplex parts of X . To extract the complex part, we use the $\langle\, . \,\rangle$ operator as follows:

$$a = \langle\, X \,\rangle \,. \tag{3}$$

To get at the uniplex part, we instead write

$$ub = \langle\, X \,\rangle_1 \quad \text{or} \quad b = \langle\, uX \,\rangle \,. \tag{4}$$

Hence

$$X = \langle X \rangle + \langle X \rangle_1 \,. \tag{5}$$

In a Clifford algebra, u would be considered a unit vector and the operator $\langle X \rangle_1$ would mean to get the vector part of X.

If A and B are any two unipodes, then

$$AB = BA \,. \tag{6}$$

The operation of 'unegation' on the unipodes is represented by the symbol $^-$. Applied to an arbitrary unipode $X = a + bu$, with a, b complex numbers, we get

$$X^- = a - bu \,. \tag{7}$$

The following theorem (though not needed for this paper) can be proved with elementary power series expansion of the exponential

$$e^{z_+ u_+ + z_- u_-} = e^{z_+} u_+ + e^{z_-} u_- . \tag{8}$$

Now, for some easy identities, most of which we won't need in this paper.

$$\begin{aligned}
u_\pm &= \tfrac{1}{2}(1 \pm u), && (9)\\
u_+ + u_- &= 1, && (10)\\
u_\pm^2 &= u_\pm, && (11)\\
u u_\pm &= \pm u_\pm, && (12)\\
u &= u_+ - u_-, && (13)\\
u_+ u_- &= 0, && (14)\\
e^{xu} &= \cosh x + u \sinh x && (15)\\
e^{-xu} &= \cosh x - u \sinh x && (16)\\
\cosh x &= \tfrac{1}{2}(e^{xu} + e^{-xu}) = \tfrac{1}{2}(e^{x} + e^{-x}) \,, && (17)\\
\sinh x &= \tfrac{1}{2}u(e^{xu} - e^{-xu}) = \tfrac{1}{2}(e^{x} - e^{-x}) \,, && (18)\\
\cosh^2 x - \sinh^2 x &= 1 \,, && (19)\\
u_\pm^- &= u_\mp, && (20)\\
u^{-1} &= u \,. && (21)
\end{aligned}$$

And, most important to our problem is the following indentity:

$$\int e^{axu} \, dx = \frac{u}{a} e^{axu} + C \,, \tag{22}$$

where C is an arbitrary complex number.

3 Getting to the integral

Once again,

$$I = \int (a \cosh x - \sinh x) \cosh bx \, dx \,. \tag{23}$$

Our first innovation is to write

$$(a \cosh x - \sinh x) = \langle\, (a-u)(\cosh x + u \sinh x) \,\rangle = \langle\, (a-u) e^{xu} \,\rangle \,. \tag{24}$$

And from (17) we can write

$$\cosh bx = \tfrac{1}{2}(e^{bxu} + e^{-bxu}) \,. \tag{25}$$

On substituting these into (23), we get

$$\begin{aligned}
I &= \tfrac{1}{2}\Big\langle\, (a-u) \int e^{xu}(e^{bxu} + e^{-bxu}) \, dx \Big\rangle \\
&= \tfrac{1}{2}\Big\langle\, (a-u) \int (e^{(1+b)xu} + e^{(1-b)xu}) \, dx \Big\rangle \\
&= \tfrac{1}{2}\Big\langle\, (a-u) \Big[\int e^{(1+b)xu} dx + \int e^{(1-b)xu} \, dx \Big] \Big\rangle \\
&= \tfrac{1}{2}\Big\langle\, (a-u) \Big[\frac{u e^{(1+b)xu}}{1+b} + \frac{u e^{(1-b)xu}}{1-b} \Big] \Big\rangle \\
&= \tfrac{1}{2}\Big\langle\, (au-1) \Big[\frac{e^{(1+b)xu}}{1+b} + \frac{e^{(1-b)xu}}{1-b} \Big] \Big\rangle \\
&= \tfrac{1}{2}\Big\langle\, (au-1) \Big[\frac{\cosh(1+b)x + u \sinh(1+b)x}{1+b} + \frac{\cosh(1-b)x + u \sinh(1-b)x}{1-b} \Big] \Big\rangle \\
&= \tfrac{1}{2}\Big\langle\, \Big[\frac{-\cosh(1+b)x + a \sinh(1+b)x}{1+b} + \frac{-\cosh(1-b)x + a \sinh(1-b)x}{1-b} \Big] \Big\rangle \,.
\end{aligned}$$

And finally, we have that

$$I = \tfrac{1}{2}\Big[\frac{a \sinh(1+b)x - \cosh(1+b)x}{1+b} + \frac{a \sinh(1-b)x - \cosh(1-b)x}{1-b} \Big] + C \,, \tag{26}$$

where C is an arbitrary complex number.

This answer can be tested by differentiating I, using that

$$\begin{aligned}
D_x \cosh x &= \sinh x \,, \\
D_x \sinh x &= \cosh x \,,
\end{aligned}$$

and that

$$\begin{aligned}
\cosh(a+b) + \cosh(a-b) &= 2 \cosh a \cosh b \,, \\
\sinh(a+b) + \sinh(a-b) &= 2 \sinh a \cosh b \,.
\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{dI}{dx} &= a\tfrac{1}{2}\big(\cosh(1+b)x + \cosh(1-b)x\big) - \tfrac{1}{2}\big(\sinh(1+b)x + \sinh(1-b)x\big) \\ &= a\tfrac{1}{2}\big(\cosh(x+bx) + \cosh(x-bx)\big) - \tfrac{1}{2}\big(\sinh(x+bx) + \sinh(x-bx)\big) \\ &= a\cosh x\cosh bx - \sinh x\cosh bx \\ &= (a\cosh x - \sinh x)\cosh bx\,. \end{aligned} \tag{27}$$

And that gives us the integrand we started with.

4 Conclusion

The unipodal algebra has its foundation in Clifford(Geometric) algebra (whose origin is in 1878), and has its roots going back to 1985 (when I used it to solve for the roots to the cubic), the name 'unipodal numbers' being coined by Garret Sobczyk in 1991. The closest thing to it, that is well-known, is the Bicomplex (commutative) algebra, which has been around much longer.

The Bicomplex number system (invented by Corrado Segre in 1892) was developed as the linear combinations of real multiples of four linearly independent numbers: Unity and two distinct imaginary numbers, typically represented by $\mathbf{I}_1$ and $\mathbf{I}_2$, and their product $\mathbf{I}_1\mathbf{I}_2$, which is the unipotent element of the system. The integral presented here could just as easily be done in the Bicomplex algebra.

Anyway, back to the unipodal algebra. We can also mix and match regular trigonometric functions in with hyperbolic functions as integrands. The reader is now asked to perform the following integral by using the unipodal algebra:

$$I = \int \cos x \cosh x\, dx\,, \tag{28}$$

where x is real.

Hint:

$$\begin{aligned} I &= \mathrm{Re}\int e^{ix}e^{xu}\,dx \\ &= \mathrm{Re}\int e^{(i+u)x}\,dx \\ &= \mathrm{Re}\left[\frac{1}{i+u}e^{(i+u)x}\right] + C \\ &= \dots \end{aligned}$$

and so forth.