Using Mathematical Induction on the AM-GM and Nicomachus's Theorems

P. Reany

March 4, 2022

Abstract

This paper uses the specialized induction techniques I developed long ago to help me solve problems that use mathematical induction. Here we solve two well-known theorems by using induction: The AM-GM theorem and Nicomachus's theorem. There are many interesting ways to prove both of these theorems that don't use induction, but this paper exists to help teach induction.

1 Introduction

The AM-GM Theorem is a relation between the arithmetic mean of a finite set of numbers to their geometric mean. Let $A = \{a_1, a_2, \dots, a_n\}$ be a list of n positive real numbers, arranged in monotonically increasing form, meaning that

$$a_1 \leq a_2 \leq \cdots \leq a_n \,. \tag{1}$$

(I include this construction because I need it for the proof I will present.) Then, the AM-GM theorem claims that

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq (a_1 a_2 \cdots a_n)^{1/n} \quad \text{for } n \geq 2 \,, \tag{2}$$

where the LHS is the arithmetic mean, and the RHS is the geometric mean.

2 Proof for special case $n = 2$

Before we prove the general case, let's prove it for the case $n = 2$, where we get the relation

$$\frac{a_1 + a_2}{2} \geq \sqrt{(a_1 a_2)} \,, \tag{3}$$

and the proof will not need induction. Without loss of generality, we can replace the above equation with

$$(a_1 + a_2)^2 \geq 4(a_1 a_2) \,, \tag{4}$$

Now, note that

$$(a_1 + a_2)^2 = a_1^2 + 2a_1a_2 + a_2^2\,,$$
$$(a_1 - a_2)^2 = a_1^2 - 2a_1a_2 + a_2^2\,.$$

If we subtract the latter of these from the former, we get that

$$(a_1 + a_2)^2 - (a_1 - a_2)^2 = 4(a_1a_2)\,. \tag{5}$$

Now, since $(a_1 - a_2)^2 \geq 0$, then, by dropping the second term on the LHS, we get that

$$(a_1 + a_2)^2 \geq 4(a_1a_2)\,, \tag{6}$$

which is equivalent to (3).

3 Proof for general case

Note: See the Appendix for my write-up on how I do mathematical induction on certain classes of problems (particularly for finite sums and for finite products).

Now to employ induction for the general case. Obviously, (2) is true for case $n = 1$, which is our base case. Next, we formulate our inductive hypothesis: We assume that (2) is true for all k and then show that, with that added assumption.

$\triangleright$ First, a lemma. Claim:

$$(a_1a_2\cdots a_{k+1})^{1/(k+1)} \geq (a_1a_2\cdots a_k)^{1/k}\,. \tag{7}$$

But this last relation is true iff

$$(a_1a_2\cdots a_{k+1}) \geq (a_1a_2\cdots a_k)^{(k+1)/k}\,, \tag{8}$$

iff

$$(a_1a_2\cdots a_k)a_{k+1} \geq (a_1a_2\cdots a_k)(a_1a_2\cdots a_k)^{1/k}\,, \tag{9}$$

iff

$$a_{k+1} \geq (a_1a_2\cdots a_k)^{1/k}\,, \tag{10}$$

iff

$$a_{k+1}^k \geq a_1a_2\cdots a_k\,, \tag{11}$$

which is true because of (1).
■

So, let's get started. For all $k \geq 1$ we assume that

$$\frac{a_1 + a_2 + \cdots + a_k}{k} \geq (a_1a_2\cdots a_k)^{1/k}\,. \tag{12}$$

Let's first 'simplify' the last equation to its equivalent form:

$$a_1 + a_2 + \cdots + a_k \geq k(a_1 a_2 \cdots a_k)^{1/k} \,. \tag{13}$$

We'll refer to this last proposition as $P(k)$. Our job now is to show that

$$a_1 + a_2 + \cdots + a_k + a_{k+1} \geq (k+1)(a_1 a_2 \cdots a_{k+1})^{1/(k+1)} \,, \tag{14}$$

which we'll refer to as $P(k+1)$.

Before continuing, some notation: Let $L(k)$ stand for the LHS of $P(k)$. Similarly, let $R(k)$ stand for the RHS of $P(k)$. Now, $P(k)$ implies $P(k+1)$ iff

$$L(k+1) - L(k) \geq R(k+1) - R(k) \,, \tag{15}$$

which is true iff

$$\begin{aligned} a_{k+1} &\geq (k+1)(a_1 a_2 \cdots a_{k+1})^{1/(k+1)} - k(a_1 a_2 \cdots a_k)^{1/k} \\ &= k[\,(a_1 a_2 \cdots a_{k+1})^{1/(k+1)} - (a_1 a_2 \cdots a_k)^{1/k}\,] + (a_1 a_2 \cdots a_{k+1})^{1/(k+1)} \\ &\geq (a_1 a_2 \cdots a_{k+1})^{1/(k+1)} \quad \text{(where we used the lemma above)} \,, \end{aligned} \tag{16}$$

which is true iff

$$a_{k+1}^{k+1} \geq a_1 a_2 \cdots a_{k+1} \,, \tag{17}$$

which is true because $a_{k+1} \geq a_\ell$ for all $\ell \leq k+1$ because of (1).
■

4 Nicomachus's Theorem

Lemma 2

$$1 + 2 + 3 + \cdots + n = \tfrac{1}{2} n(n+1) \equiv \Omega_n \,. \tag{18}$$

If you want, you can prove this lemma by the techniques shown in this paper.

Nicomachus's Theorem: For any $n \geq 1$

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2 = \Omega_n^2 \,. \tag{19}$$

Proof: For $n = 1$ it is clearly true. Next, we assume the result for arbitrary k and then show that that assumption can establish the result for $k+1$. Thus, we assume that

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = (1 + 2 + 3 + \cdots + k)^2 = \Omega_k^2 \,, \tag{20}$$

for arbitrary k. So, $P(k+1)$ is true iff

$$L(k+1) - L(k) = R(k+1) - R(k) \,, \tag{21}$$

or

$$
\begin{aligned}
(k+1)^3 &= \Omega_{k+1}^2 - \Omega_k^2 \\
&= (\Omega_k + k + 1)^2 - \Omega_k^2 \\
&= 2(k+1)\Omega_k + (k+1)^2 \\
&= 2(k+1)[\tfrac{1}{2}k(k+1)] + (k+1)^2 \\
&= (k+1)(k+1)^2 \\
&= (k+1)^3 \,,
\end{aligned} \tag{22}
$$

which is true.

5 Conclusion

So, was our proof by induction of the AM-GM inequality really a simple way to proceed? Well, it must be admitted that there was a lot of computation involved; however, on the positive side, the way to proceed from start to finish seemed quite straightforward to me.

6 Appendix: Abstract mathematical induction

Given two functions $L(n)$ and $R(n)$, whose domains are the integers greater than or equal to some integer m, that is, the base case, which ios often zero or one, if we propose that

$$L(n) = R(n) \,, \tag{23}$$

where "$L(n)$" is the left expression and "$R(n)$" is the right expression," we can, in principle, test the validity of (23) for a finite set of integers starting at the smallest integer m; but if we propose that (23) is true for all integers n such that $n \geq m$, we must often rely on the principle of mathematical induction. A standard approach to mathematical induction is the propositional approach described below.[1]

Let $P(n)$ be the proposition that the algebraic sentence (23) is true for all integers n such that $n \geq m$. Let $P(m)$ be the validity of the proposition that $L(m) = R(m)$, similarly, let $P(k)$ be the validity of $L(k) = R(k)$. Then, to prove that $P(n)$ is true by mathematical induction, we must

1) show that $P(m)$ is true
2) assume $P(k)$ is true for some arbitrary $k \geq m$
3) show, on the basis of 2), that $P(k+1)$ is true

We can now associate an algebraic interpretation of the above steps. As before, $P(m)$ is true if and only if $L(m) = R(m)$. Assuming that $P(k)$ is true is logically equivalent to assuming that $L(k) = R(k)$. Then, to show that $P(k+1)$ is true we need to show that

$$L(k+1) = R(k+1) \tag{24}$$

given that $L(k) = R(k)$.

Up to this point we have been quite general, but now we consider the special class of problems where $L(n)$ is a finite sum and $R(n)$ is a closed form. Let

$$L(n) = a_m + a_{m+1} + a_{m+2} + \cdots + a_n \, , \tag{25}$$

and let $R(n)$ be any closed form such that $P(n)$ is the proposition that $L(n) = R(n)$ for all $n \geq m$. Considering only those cases where $P(m)$ is true, all we need do is to prove $P(n)$ is to assume $P(k)$ to prove $P(k+1)$

According to our procedure, we assume $L(k) = R(k)$, then $L(k+1) = R(k+1)$ if and only if

$$R(k+1) = R(k) + [L(k+1) - L(k)] = R(k) + a_{k+1} \, ,$$

in which case then $P(k+1)$ is true. We have just proved the following theorem:

[1] Saracino, D. 1980. *Abstract Algebra: A First Course.* Massachusetts: Addison-Wesley Publishing Company.
[2] Hillman, A. P. and Alexanderson, G. L. 1978. *A First Course in Abstract Algebra*, 2nd ed. Belmont, California: Wadsforth Publishing Company.
[3] Swokowski, E. 1984. *Calculus with Analytic Geometry.* 3rd ed. pp. A1–A5. Massachusetts: Prindle, Weber, and Schmidt Publishers.
[4] Polya, G. 1965. *Induction and Analogy in Mathematics*, 5th ed. pp. 108–110. Princeton University Press.