Chebyshev Polynomials, Fibonacci and Lucas Numbers in the Unipodal Algebra

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February 24, 2020

Abstract

The Chebyshev polynomials of first and second kinds are revealed as solutions to the unipodal equation, $X^n = b$, effecting a simple unification of the two kinds. Then the differential equations they satisfy are revealed from a single unipodal differential equation. Many of the identities these polynomials obey are deduced. The well-known connection to the Fibonacci and Lucas numbers is exploited, while demonstrating two distinct unipodal formulations that efficiently unite them. Also, the unification of the Chebyshev polynomials of first and second kinds is alternatively made by a single unipodal generating function.

Chebyshev Polynomials, Fibonacci numbers, Lucas Numbers, Unipodal algebra, generating functions.

1 Introduction

Contrary to how this subject is usually introduced, in this paper the Chebyshev polynomials (of both first and second kinds) are revealed as solutions to the simple unipodal algebraic equation

$$X^n = b\,, \tag{1}$$

demonstrating a natural unification of the Chebyshev polynomials of both kinds, which seems to be rather lacking in the literature.[1] Then the differential equations they satisfy are revealed. In the lemmas, many of the identities that these polynomials obey are deduced. After that, the relevance of Chebyshev to the Fibonacci and Lucas numbers is made, and, demonstrating how the Fibonacci and Lucas numbers can be embedded into the unipodal algebra in two different ways, simplifying some proofs to near triviality.

The standard approaches to this subject utilize trigonometric and inverse trigonometric functions, generating functions, mathematical induction, and even matrices to ferret-out the properties of the Chebyshev polynomials, Fibonacci

[1] For a graphic on how these polynomials are united, see Figure 1, page 15.

and Lucas numbers. But in this paper, we will need little use of those devices, relying more often on the power of the unipodal algebra directly to discover the hidden treasures contained in the simple unipodal equation (1).[2]

As to the importance of Chebyshev polynomials, I'll let experts in the field, Benjamin & Walton [3], describe it:

> It's hard to avoid Chebyshev polynomials. They appear in just about every branch of mathematics, including geometry, combinatorics, number theory, differential equations, approximation theory, numerical analysis, and statistics.

Perhaps the ubiquity of the Chebyshev polynomials, implied in this quote, results from the shear simplicity of Eq. (1) over the unipodal numbers.

A few years after the publishing of the seminal paper, "Unipodal Algebra and Roots of Polynomials" ([4], 1991) that I particpated in, I published the paper,"Unipodal Algebra Reopens the Theory of Algebraic Equations" ([11], 1994), in which I stumbled upon the Chebyshev polynomials, without realizing it at the time. The recurrence relation I had discovered was

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)\,, \quad \text{for } n \geq 2\,, \tag{2}$$

with $T_0 = 1$ and $T_1 = x$ (though I used a generic f_n in place of T_n). This current paper will further develop the theory of Chebyshev polynomials in the unipodal algebra.

In the literature on this subject, the starting points are many and varied. Some start with the recurrence relation above (2). Some start with the Chebyshev differential equation, to be revealed later. Others, such as those demonstrating a connection between the Chebyshev, Fibonacci, and Lucas polynomials, start with the following equations:

$$T_n(x) = \tfrac{1}{2}\big[(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n\big]\,, \tag{3a}$$

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}\big[(x+\sqrt{x^2-1})^{n+1} - (x-\sqrt{x^2-1})^{n+1}\big]\,, \tag{3b}$$

$$L_n(x) = \frac{1}{2^n}\big[(x+\sqrt{x^2+4})^n + (x-\sqrt{x^2+4})^n\big]\,, \tag{3c}$$

$$F_n(x) = \frac{1}{2^n\sqrt{x^2+4}}\big[(x+\sqrt{x^2+4})^n - (x-\sqrt{x^2+4})^n\big]\,, \tag{3d}$$

where the first is the Chebyshev polynomials of the first kind, the second is the Chebyshev polynomials of the second kind, and the third is the Lucas polynomials, and the last is the Fibonacci polynomials. For a reference, see [6]. Others

[2]Casting the Chebyshev polynomials, along with the Fibonacci and Lucas numbers, into the unipodal algebra has the added benefit that students proficient in high school algebra, but lacking in knowledge of trigonometry, induction, and generating functions, can, after a brief introduction to the unipodal algebra, still grasp a great deal of the mathematical content of this paper. Another benefit should be the easier manipulation of these polynomials and functions in computer algebra systems.

start with the generating function for the Gegenbauer polynomials (see Arfken et al [1]). Still others start with the Rodrigues formulation of the Chebyshev polynomials (see [10]).

Our starting point is (1) together with one simple constraint:

$$XX^- = 1\,, \tag{4}$$

where X^- is the unegation of the unipodal number X (to be defined shortly). From these two equations we will develop a unified theory of Chebyshev polynomials of first and second kinds. We will soon see that their optimal optimal 'polynomial' forms (in variables X and X^-) are:

$$T_n(x) = \tfrac{1}{2}\left[X^n + X^{-n}\right], \tag{5a}$$

$$U_n(x) = \frac{u}{2X_1}\left[X^{n+1} - X^{-(n+1)}\right], \tag{5b}$$

where $X = X_0 + X_1 u$ and X_0 has been replaced by x.

Now, before we begin a short introduction to the basic properties of the unipodal algebra, a word of comparison to similar algebras is worth making. The unipodal numbers are much older than I realized when paper [4] came out in 1991, in which the unipodal numbers were treated as the simplest possible Clifford algebra over the complex scalars. There, the unipodal numbers were introduced as an extention of the complex numbers by adding to them a 'unipotent' number u (which in a Clifford algebra is just a unit vector) with the properties that it is **not** a complex number but it squares to unity (in other words, it is not ± 1), and it commutes with all complex numbers.[3]

There are other number systems that have been around longer than the unipodal number system (by name, anyway), but are isomorphic to it. Among them are the *bicomplex numbers*, which have been around since at least the late nineteenth century, and today it has an abundant body of papers and books attached to them ([5], [12], [9], and [8], to name a few). But for the benefit of those who read the literature of both systems, I want to explain why the 'look and feel' of these two isomorphic number systems are so different. The bicomplex numbers are treated as the commutative extension of the complex numbers by a distinct unit imaginary (and often treated as a 4-dimensional vector space over the real numbers); whereas, the unipodal numbers are treated as the extension of the complex numbers by a unipotent number (and treated as a 2-dimensional vector space over the complex numbers).

2 Unipodal Powers and Exponentials

We now briefly review the unipodal algebra $\mathcal{U}$ (a one-dimensional Clifford algebra over the complex scalars). Let a_0 and a_1 be arbitrary complex numbers. $\mathcal{U}$ is formed by the set of all linear combinations $a_0 + a_1 u$, where u is a noncomplex

[3] Wikipedia defines a *unipotent* number differently than it is defined in this paper.

number that commutes with complex numbers, and has the additional property that $u^2 = 1$. Thus, a typical unipodal number looks like

$$a = a_0 + a_1 u \,. \tag{6}$$

We call a_0 and a_1 the *complex* and *uniplex* parts, respectively, of a. For an arbitrary $\mathcal{U}$ number, or *unipode* a, its complex part is also denoted by $<a> = <a>_0 = a_0$ and its uniplex part by $a_1 = <au>$. The two elements $\{1, u\}$ form the *standard basis* to $\mathcal{U}$. In Clifford algebra terminology, a_0 would be called the *scalar* part and $a_1 u$ would be called the *vector* part of the unipode, though in unipodal speak, we'll refer to $a_1 u$ as the *unisor* part of the unipode.

We'll represent complex conjugation by the operator $(^*)$, so that $i^* = -i$, though complex conjugation will be used surprisingly little in this paper. Under complex conjugation, $u^* = u$. We now define a means to conjugate u, while leaving complex numbers unchanged. Let $a = a_0 + u a_1$ be any unipode, then '*a unegate*', denoted as a^-, is defined by $a^- = a_0 - u a_1$. So that when $a_0 = 0$ and $a_1 = 1$, $u^- = -u$. Thus, the complex part of a unipode is that part which remains unchanged when the unipode is unegated. For any unipode a, its *modulus* is defined as $aa^- = a_0^2 - a_1^2$, which is a complex number. In other words, $aa^- = <aa^->$. It's a simple matter to show that for arbitrary unipode a, $(a^-)^- = a$.

A unipode is said to be *singular* if its modulus is zero. A *nonsingular* unipode a has inverse

$$a^{-1} = \frac{a^-}{aa^-} \,, \tag{7}$$

which is easy to verify. If a is both nonsingular and normalized by $aa^- = 1$, then we have the most fortuitous relation,

$$a^{-1} = a^- \,, \tag{8}$$

from which we have the further good fortune of:

$$(a^-)^n = (a^{-1})^n = a^{-n} \tag{9}$$

Reader Exercise: Prove the identity for arbitrary unipodes a, b:

$$(ab)^- = a^- b^- \,. \tag{10}$$

The unipodal algebra has another basis, the *idempotent basis* $\{u_+, u_-\}$, defined by[4]

$$u_\pm = \tfrac{1}{2}(1 \pm u) \,. \tag{11}$$

The great advantage of an *idempotent* is that it squares to itself! It's easy to check that $u_\pm^2 = u_\pm$. The other important properties of these numbers are that $u_+ + u_- = 1$, $u_+ u_- = 0$, and $u_\pm^- = u_\mp$.

[4]Far from losing power by adding in zero-divisors, the unipodal algebra retains the full power of the complex numbers because, whether in the standard basis or the idempotent basis, the unipodal numbers have two copies of the complex numbers as their components.

Let x be an arbitrary unipode, then the *exponential* of x is defined by

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{12}$$

We define x^0 to be unity for all nonzero unipodes.

Now, every unipode $a = a_0 + a_1 u$ can be expressed in idempotent basis as

$$a_0 + a_1 u = a_+ u_+ + a_- u_- \,, \tag{13}$$

where $a_\pm = a_0 \pm a_1$. Reader exercise: Express a^n in idempotent basis for n a positive integer.

The following represents some equivalent forms for exponentials.

$$\begin{aligned} e^{a_0 + a_1 u} &= e^{a_+ u_+ + a_- u_-} = e^{a_+} u_+ + e^{a_-} u_- \\ &= \tfrac{1}{2}(e^{a_+} + e^{a_-}) + \tfrac{1}{2}(e^{a_+} - e^{a_-})u \\ &= \tfrac{1}{2} e^{a_0}(e^{a_1} + e^{-a_1}) + \tfrac{1}{2} e^{a_0}(e^{a_1} - e^{-a_1})u \\ &= e^{a_0} \cosh a_1 + u e^{a_0} \sinh a_1 = e^{a_0} e^{u a_1} \,. \end{aligned} \tag{14}$$

This last set of equalities is one of the best examples I can think of to demonstrate the advantages of switching between the standard basis and the idempotent basis. Deriving $e^{a_+} u_+ + e^{a_-} u_-$ from $e^{a_+ u_+ + a_- u_-}$ is a trivial exercise in using the definition of the power series expansion for e^x and the fact that the idempotents are mutually annihilating. This kind of efficiency is impossible when one is restricted to the complex numbers.

For use later on, note simpler forms as

$$e^{\theta u} = \cosh\, \theta + u \sinh\, \theta \quad \text{and} \tag{15a}$$

$$e^{n\theta u} = \cosh\, n\theta + u \sinh\, n\theta \,, \tag{15b}$$

where θ would typically be real-valued, but it could be complex-valued. When it's pure imaginary, such as if we set $\theta = i\omega$, where ω is real-valued, we get

$$e^{i\omega u} = \cosh i\omega + u \sinh i\omega = \cos\omega + iu \sin\omega \quad \text{and} \tag{16a}$$

$$e^{ni\omega u} = \cosh ni\omega + u \sinh ni\omega = \cos n\omega + iu \sin n\omega \,. \tag{16b}$$

It's easy to establish that $\cosh i\omega = \cos\omega$ and $\sinh i\omega = i \sin\omega$ by expanding cosh and sinh in terms of their power series.

Typically, the Chebyshev polynomials are developed in terms of trigonometric functions, but I prefer to develop them in terms of hyperbolic functions. First, because these functions seem to me to be more natural in the unipodal algebra, and, second, because they have 'better' derivative properties, such as never producing a negative sign on differentiation. For example,

$$\frac{d}{dx} \cosh x = \sinh x \quad \text{and} \quad \frac{d}{dx} \sinh x = \cosh x \,. \tag{17}$$

3 The Unipodal Foundation to the Chebyshev Polynomials

We now have enough basics of the unipodal algebra to formally present the **unipodal version of the Chebyshev polynomials.** As stated before, we begin with

$$X^n = b\,, \tag{18}$$

where $X = X_0 + X_1 u$, and where X_0 and X_1 are, generally, complex-valued. And $b = b_0 + b_1 u$, where b_0 and b_1 are also, generally, complex-valued. Now, we add just one constraint:

$$XX^- = 1\,, \tag{19}$$

which is a profitable way to map unipodal equations into complex equations. Now, (19) induces the following constraint on b:

$$bb^- = 1\,, \tag{20}$$

and, remarkably, we're done! By the way, we shall make full use of the facts that $X^- = X^{-1}$ and $b^- = b^{-1}$, and the result of combining Equations (18) and (19) is to give us the unipodal equivalence to de Moivre's Theorem in the complex numbers.

Our first task is to show that $\frac{1}{2}(X^n + X^{-n})$ is pure complex. A unipode is pure complex if it is equal to its own unegate. Since

$$\tfrac{1}{2}(X^n + X^{-n})^- = \tfrac{1}{2}(X^{-n} + X^n) = \tfrac{1}{2}(X^n + X^{-n})\,, \tag{21}$$

then $\frac{1}{2}(X^n + X^{-n})$ is pure complex.

We have a simple way to write X^n in terms of its complex and unisor parts through the following identity:

$$X^n = \tfrac{1}{2}(X^n + X^{-n}) + \tfrac{1}{2}(X^n - X^{-n})\,. \tag{22a}$$
$$= b_0 + ub_1\,. \tag{22b}$$

Therefore

$$b_0 = \tfrac{1}{2}(X^n + X^{-n})\,, \tag{23a}$$
$$b_1 = \frac{u}{2}(X^n - X^{-n})\,. \tag{23b}$$

The expression $X^n - X^{-n}$ in (23b) will be shown to be a significant part of the definition of the Chebyshev polynomials of the second kind. Now we'll prove that $X^n - X^{-n}$ is pure unisor. A unipode is pure unisor if it changes sign under unegation. We note that

$$[X^n - X^{-n}]^- = X^{-n} - X^n \ = -[X^n - X^{-n}]\,, \tag{24}$$

hence that $X^n - X^{-n}$ is pure unisor.

4 Converting to the Idempotent Basis

The last thing to accomplish is to produce a few simple but useful formulas based on writing unipodes in the idempotent basis. We start with converting the unipodal equation (18) to its idempotent basis form:

$$(X_0 + X_1 u)^n = b_0 + b_1 u \tag{25a}$$
$$(X_+ u_+ + X_- u_-)^n = b_+ u_+ + b_- u_- \tag{25b}$$
$$X_+^n u_+ + X_-^n u_- = b_+ u_+ + b_- u_- \,, \tag{25c}$$

where

$$X_+ = X_0 + X_1 \quad \text{and} \quad X_- = X_0 - X_1 \,, \tag{26}$$

and

$$b_+ = b_0 + b_1 \quad \text{and} \quad b_- = b_0 - b_1 \,. \tag{27}$$

On equating coefficients of like terms, we get from (25c) that

$$X_+^n = b_+ \quad \text{and} \quad X_-^n = b_- \,. \tag{28}$$

Now, employing (11), we can recast (25c) back into standard form:

$$\tfrac{1}{2}[X_+^n + X_-^n] + \tfrac{1}{2}[X_+^n - X_-^n]u = b_0 + b_1 u \,. \tag{29}$$

Hence,

$$b_0 = \tfrac{1}{2}[X_+^n + X_-^n] \,, \tag{30a}$$
$$b_1 = \tfrac{1}{2}[X_+^n - X_-^n] \,. \tag{30b}$$

The following relations are useful when X is in the idempotent basis:

$$X = X_+ u_+ + X_- u_- \,, \tag{31a}$$
$$X^- = X_- u_+ + X_+ u_- \,, \tag{31b}$$
$$X^n = X_+^n u_+ + X_-^n u_- \,, \tag{31c}$$
$$X^{-n} = X_-^n u_+ + X_+^n u_- \,. \tag{31d}$$

From the last set of equations, it's easy to show that

$$X^n + X^{-n} = X_+^n + X_-^n \,, \tag{32a}$$
$$X^n - X^{-n} = (X_+^n - X_-^n)u \,. \tag{32b}$$

Reader exercise: Show that a sufficient condition that unipode X has an inverse is that $X_+ X_- \neq 0$.

5 Developing the Chebyshev Polynomials of the First Kind

As we saw, the whole trick to getting the Chebyshev polynomials out of (1) is to set the *modulus* of X to unity by condition (19), and as a consequence we get,

$$X^- = X^{-1}\,, \tag{33a}$$

$$b^- = b^{-1}\,. \tag{33b}$$

There are two ways we can proceed. First, we could parameterize X and b in terms of a real (or even complex) variable θ and derive

$$X = e^{u\theta}, \qquad b^{(n)} = b_0^{(n)} + u b_1^{(n)} = e^{nu\theta}\,, \tag{34}$$

where $b_0^{(n)} = \cosh n\theta$, $b_1^{(n)} = \sinh n\theta$, and work directly with hyperbolic functions. Proceeding this way, we can benefit from the simple forms presented in Equations (15a) and (15b). Thus,

$$XX^{-1} = XX^- = X_+X_- = X_0^2 - X_1^2 = \cosh^2\theta - \sinh^2\theta = 1\,, \tag{35}$$

and

$$bb^{-1} = bb^- = b_+b_- = b_0^2 - b_1^2 = \cosh^2 n\theta - \sinh^2 n\theta = 1\,. \tag{36}$$

The second way to proceed is in a more algebraic fashion in terms of polynomials in the unipodal variables X and X^-. To that end, let's set

$$T_n(X_0) \equiv b_0 = \langle X^n \rangle\,, \tag{37}$$

where $\langle X^n \rangle$ is the complex part of X^n.[5] When we adopt the angular parameterization in (34), Eq. (37) has the immediate corollary

$$T_n(\cosh\theta) = \cosh n\theta\,, \tag{38}$$

which suggests that we're on the right track of establishing that (37) is the correct definition for the Chebyshev polynomials of the first kind. We can convert to the conventional version of these polynomials, which use circular trigonometric functions, by the variable substitution $\theta \to i\alpha$ (where α is a real variable), to get

$$T_n(\cos\alpha) = \cos n\alpha\,. \tag{39}$$

Using (23a) and (37), we have a closed form in unipodal variables for the Chebyshev polynomials of the first kind,

$$T_n(X_0) \equiv \tfrac{1}{2}(X^n + X^{-n})\,, \tag{40}$$

[5] It's standard to use $T_n(\ )$ for the Chebyshev polynomials of the first kind.

and we'll prove this definition to be correct by showing a) that it satifies the appropriate recurrence relation and b) that it has the appropriate first two polynomial values of $T_0(X_0) = 1$ and $T_1(X_0) = X_0$.[6]

Proof of a): Multiplying (40) across by

$$2X_0 = X + X^- = X + X^{-1}\,, \tag{41}$$

gives

$$\begin{aligned} 2X_0T_n(X_0) &= (X + X^{-1})\tfrac{1}{2}(X^n + X^{-n}) \\ &= \tfrac{1}{2}[X^{n+1} + X^{-n+1} + X^{n-1} + X^{-(n+1)}] \\ &= T_{n+1}(X_0) + T_{n-1}(X_0)\,. \end{aligned} \tag{42}$$

On rearranging, we get,

$$T_{n+1}(X_0) = 2X_0T_n(X_0) - T_{n-1}(X_0) \quad \text{for} \quad n \geq 1\,. \tag{43}$$

This last equation matches with Eq. (2) by letting $n \to n - 1$.

Proof of b): That $T_0(X_0) = 1$ and $T_1(X_0) = X_0$ follow immediately from Eq. (40). Hence, we have proved that $T_n(X_0)$ represents the Chebyshev polynomials of the first kind.

We can let the index of T be negative, but to be consistent with (40), we must have that

$$T_{-n}(X_0) = \tfrac{1}{2}(X^{-n} + X^n) = T_n(X_0)\,. \tag{44}$$

An immediate corollary from (40) is that $T_n(1) = 1$. (Hint: Since $X_0^2 - X_1^2 = 1$, $X_0 = 1$ implies that $X_1 = 0$.) Similarly, $T_n(-1) = (-1)^n$. Proof: (Again, $X_1 = 0$, etc.)

$$T_n(-1) = \tfrac{1}{2}[(-1 + 0u)^n + (-1 + 0u)^{-n}] = (-1)^n\,. \tag{45}$$

Another identity is $T_{2n}(0) = (-1)^n$. Proof: $X_0 = 0$ implies $X_1 = \pm i$, and so

$$T_{2n}(0) = \tfrac{1}{2}\{[0 + (\pm iu)]^{2n} + [(0 + (\pm iu))^-]^{2n}\} = \tfrac{1}{2}[(i^2)^n + [(i^2)^n] = (-1)^n\,. \tag{46}$$

Similarly, $T_{2n+1}(0) = 0$. Proof: Again, $X_0 = 0$ implies $X_1 = \pm i$ and

$$\begin{aligned} T_{2n+1}(0) &= \tfrac{1}{2}[(\pm iu)^{2n+1} + ((\pm iu)^-)^{2n+1}] \\ &= \tfrac{1}{2}[(\pm iu)^{2n}(\pm iu) + (\mp iu)^{2n}(\mp iu)] \\ &\tfrac{1}{2}(-1)^n[(\pm iu) + (\mp iu)] = 0\,. \end{aligned} \tag{47}$$

Also, it's not difficult to show that for all nonnegative integers n, $T_n(X_0)$ is an nth degree polynomial in X_0. The proof is left to the reader.

[6] The Chebyshev polynomials of the first and second kinds have the same recurrence relation (as we'll see later). However, the Chebyshev polynomials of the second kind distinguish themselves from the first kind by beginn ing with $U_0(X_0) = 1$ and $U_1(X_0) = 2X_0$.

A useful alternative formulation for $T_n(X_0)$ is given by

$$T_n(x) = \tfrac{1}{2}[X_+^n + X_-^n]\,, \tag{48}$$

which can be proved using (32a).

As an example calculation using (40) for $n = 4$, we get

$$\begin{aligned} T_4(X_0) &= \tfrac{1}{2}(X^4 + X^{-4}) \\ &= (1/2)[(X_0 + uX_1)^4 + (X_0 - uX_1)^4] \\ &= X_0^4 + 6X_0^2X_1^2 + X_1^4 \\ &= 8X_0^4 - 8X_0^2 + 1\,. \end{aligned} \tag{49}$$

Using (40), we can calculate the first five polynomial expressions:

$$\begin{aligned} &n = 0, \quad T_0(X_0) = 1\,, \\ &n = 1, \quad T_1(X_0) = X_0\,, \\ &n = 2, \quad T_2(X_0) = 2X_0^2 - 1\,, \\ &n = 3, \quad T_3(X_0) = 4X_0^3 - 3X_0\,, \\ &n = 4, \quad T_3(X_0) = 8X_0^4 - 8X_0^2 + 1\,. \end{aligned} \tag{50}$$

As an example of what these polynomials can be used for, remembering from (34) that $X_0 = \cosh\theta$, and $b_0^{(n)} = \cosh n\theta$, and plugging in these values into the formula for, say, T_3, where $b_0^{(3)} = T_3(X_0)$, we get

$$\cosh 3\theta = 4\cosh^3\theta - 3\cosh\theta\,. \tag{51}$$

Using (40) we can convert to a more 'complex variable' version of T_n,[7] given by:

$$T_n\big(\tfrac{z + \frac{1}{z}}{2}\big) = \tfrac{1}{2}\big(z^n + \frac{1}{z^n}\big)\,. \tag{52}$$

We already know from (41) that we can write (40) as

$$T_n\Big(\frac{X + \dfrac{1}{X}}{2}\Big) = \tfrac{1}{2}\big(X^n + \frac{1}{X^n}\big)\,. \tag{53}$$

And now, to the proof of (52):

Let x and y be real numbers. Letting $X_0 \to x$ and $X_1 \to iy$, we can write

$$X = x + jy\,, \tag{54}$$

where $j \equiv iu$ and therefore $j^2 = -1$. However, unegation on j acts like complex conjugation, i.e., $j^- = -j$. Also, we have the constraint $XX^- = 1$ to deal with. Thus,

$$x^2 + y^2 = 1\,. \tag{55}$$

[7] See [2], p. 2158 for a reference.

Letting $z = X$, we see that the set of z is the locus of points on the unit circle in the complex z-plane, with unit imaginary j. This then justifies Eq. (52), which is just a special case of (40).

6 The Chebyshev Differential Equation of the First Kind

The simplest second-order linear differential equation I can find in terms of either $T_n(X_0)$ or $b_0^{(n)}$, keeping in mind (17), is

$$\frac{d^2 b_0^{(n)}}{d\theta^2} - n^2 b_0^{(n)} = 0\,, \tag{56}$$

where the variables X_0, X_1, and θ are restricted to real values. Actually, we could start this off with the more unified unipodal equation

$$\frac{d^2 b}{d\theta^2} - n^2 b = 0\,, \tag{57}$$

where $b = e^{nu\theta} = b^{(n)}$, but, because of subtleties arising from defining the Chebyshev polynomials of the second kind in terms of $b_1^{(n)}$, I decided to solve for the components of (57) separately, starting with $b_0^{(n)}$, and then combine them later to produce (57).

Now, since the goal is to find a linear differential equation in $T_n(X_0)$, the way from here is clear: If, in (56), we replace $b_0^{(n)}$ by $T_n(X_0)$, we must also replace the derivative by θ by the corresponding derivative by X_0.

Implicit differentiation of $X_1^2 = X_0^2 - 1$ by X_0 gives us

$$\frac{dX_1}{dX_0} = \frac{X_0}{X_1}\,. \tag{58}$$

Since $X_0 = \cosh\theta$, then

$$\frac{dX_0}{d\theta} = \sinh\theta = X_1\,, \tag{59}$$

and so

$$\frac{d}{d\theta} = \frac{dX_0}{d\theta}\frac{d}{dX_0} = X_1\frac{d}{dX_0}\,. \tag{60}$$

Therefore,

$$\frac{d^2}{d\theta^2} = X_0\frac{d}{dX_0} + X_1^2\frac{d^2}{dX_0^2}\,. \tag{61}$$

Then, substituting this result into (56) with $b_0^{(n)} \to T_n(X_0)$, we get

$$(1 - X_0^2)T_n''(X_0) - X_0 T_n'(X_0) + n^2 T_n(X_0) = 0\,. \tag{62}$$

We can make this equation look more familiar by replacing X_0 by x and $T_n(X_0)$ by y, to get

$$(1 - x^2)y'' - xy' + n^2 y = 0\,, \tag{63}$$

with solution $y = T_n(x)$, and this is the familiar differential equation for the Chebyshev polynomials of the first kind.

7 The Chebyshev Polynomials of the Second Kind

Returning to (23b), we have

$$ub_1^{(n)} = \tfrac{1}{2}(X^n - X^{-n})\,. \tag{64}$$

Proceeding in a manner similar to the substitution we made to replace $b_0^{(n)}$ by $T_n(X_0)$, we set

$$f_n(X_1) \equiv b_1^{(n)}\,, \tag{65}$$

where f_n, although not the Chebyshev polynomials of the second kind we seek, is a convenient precursor to it.

Thus,

$$uf_n = \tfrac{1}{2}(X^n - X^{-n})\,. \tag{66}$$

Then, as before,

$$2X_0 = X + X^-\,, \tag{67}$$

and multiplying these last two equations together yields

$$\begin{aligned} 2uX_0 f_n &= (X + X^-)\tfrac{1}{2}(X^n - X^{-n}) \\ &= \tfrac{1}{2}[(X^{n+1} - X^{-(n+1)}) + (-X^{-(n-1)} + X^{n-1})] \\ &= uf_{n+1} + uf_{n-1}\,. \end{aligned} \tag{68}$$

On rearranging, and letting $n \to n-1$, we get the recurrence relation:

$$f_n(X_1) = 2X_0 f_{n-1}(X_1) - f_{n-2}(X_1) \quad \text{for} \quad n \geq 2\,. \tag{69}$$

Setting $n = 0, 1, 2, 3$, in turn, in (66), with $X = X_0 + X_1 u$, and with $X^{-n} = (X^-)^n$, gives us, respectively:

$$\begin{aligned} &n = 0, \quad f_0 = 0\,, \\ &n = 1, \quad f_1 = X_1\,, \\ &n = 2, \quad f_2 = 2X_0 X_1\,, \\ &n = 3, \quad f_3 = 4X_0^2 X_1 - X_1\,, \end{aligned} \tag{70}$$

where for f_3 we used that $X_1^2 = X_0^2 - 1$. Clearly, we have here the possibility of constructing a second sequence of Chebyshev polynomials, similar to, but not equal to, the Chebyshev polynomials of the first kind. But we shouldn't start this new sequence at $f_0 = 0$.

We can define a second set of Chebyshev polynomials U_n (of the 'second kind') by the defining relation:

$$U_{n-1}(X_0) \equiv \frac{f_n}{X_1} \quad \text{for} \quad n \geq 1\,, \tag{71}$$

since f_n is evenly divisible by X_1 for all $n \geq 1$, which is provable by a simple induction proof. The motivation for dividing through by X_1 is straightforward.

If U_{n-1} is to be a polynomial in X_0, it cannot contain odd powers of X_1, which would introduce square roots, i.e., $\pm\sqrt{X_0^2-1}$. Dividing f_n by X_1 will convert odd powers of X_1 to even powers.

Using (65), we can write (71) in the form

$$U_{n-1}(X_0) \equiv \frac{b_1^{(n)}}{X_1} = \frac{u(X^n - X^{-n})}{2X_1} \quad \text{for} \quad n \geq 1\,, \tag{72}$$

or in the equivalent form

$$U_n(X_0) = \frac{u}{2X_1}(X^{n+1} - X^{-(n+1)}) \quad \text{for} \quad n \geq 0\,. \tag{73}$$

It's not difficult to show that the extension of U_n to negative indices requires that

$$U_{-n}(X_0) = -U_{n-2}(X_0) \quad \text{for} \quad n \geq 1\,. \tag{74}$$

The first few Chebyshev polynomials of the second kind are:

$$\begin{aligned} U_{-1} &= 0\,, \\ U_0 &= 1\,, \\ U_1 &= 2X_0\,, \\ U_2 &= 4X_0^2 - 1\,, \\ U_3 &= 8X_0^3 - 4X_0\,. \end{aligned} \tag{75}$$

Using (65) to eliminate f_n in (69), letting $n \to n+1$, and simplifying, leaves us with the recurrence relation for $U_n(X_0)$:

$$U_{n+1}(X_0) = 2X_0U_n(X_0) - U_{n-1}(X_0) \quad \text{for} \quad n \geq 1\,, \tag{76}$$

which is the same recurrence relation obeyed by $T_n(X_0)$.

With the aid of (32b) we have the alternative forms

$$U_{n-1}(X_0) = \frac{1}{2X_1}(X_+^n - X_-^n) \quad \text{for} \quad n \geq 1\,, \tag{77a}$$

or, incrementing n:

$$U_n(X_0) = \frac{1}{2X_1}(X_+^{n+1} - X_-^{n+1}) \quad \text{for} \quad n \geq 0\,. \tag{77b}$$

This last form is instrumental in giving an elegant proof of the composition identity:

$$U_n(T_m(x)) = \frac{U_{m(n+1)-1}(x)}{U_{m-1}(x)}\,, \tag{78}$$

to be presented later.

Reader Exercise: Prove the identity (3b) by use of (77b).

8 What to Do When $X_1 = 0$

For most symbolic uses of $U_n(X_0)$ presented in the above definitions and their alternative versions, having X_1 in the denominator, poses no problem, until $X_1 = 0$, which occurs when $X_0 = \pm 1$. We've already shown that the X_1 in the denominator can be canceled out. Doing so will produce the following formula for the Chebyshev polynomials of the second kind:

$$U_{n-1}(X_0) = \sum_{\ell=0,1,\ldots}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2\ell-1} X_0^{n-2\ell+1} (X_0^2 - 1)^{\ell-1} \,. \tag{79}$$

Before beginning the proof proper, a couple notes are useful. First, note that for any nonnegative integer k:

$$[1 - (-1)^k] u^k = \begin{cases} 0 \,, & \text{for } k \text{ even,} \\ 2u \,, & \text{for } k \text{ odd.} \end{cases} \tag{80}$$

Second, we introduce the floor function $\lfloor \cdot \rfloor$, which operates on real numbers and returns the largest integer less than or equal to its argument.

Proof: We expand both X^n and $X^{-n} = (X^-)^n$ with the binomial formula.

$$\begin{aligned} U_{n-1}(X_0) &= \frac{u}{2X_1} [X^n - (X^-)^n] \\ &= \frac{u}{2X_1} [(X_0 + X_1 u)^n - (X_0 - X_1 u)^n] \\ &= \frac{u}{2X_1} \Big[\sum_{k=0}^{n} \binom{n}{k} X_0^{n-k} X_1^k [1 - (-1)^k u^k] \Big] \\ &= \sum_{k=1,3,\ldots}^{n} \binom{n}{k} X_0^{n-k} X_1^{k-1} \\ &= \sum_{\ell=1,2\ldots}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2\ell-1} X_0^{n-2\ell+1} X_1^{2\ell-2} \\ &= \sum_{\ell=1,2,\ldots}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2\ell-1} X_0^{n-2\ell+1} (X_0^2 - 1)^{\ell-1} \,. \end{aligned} \tag{81}$$

Thus, for $X_0 = 1$ and thus $X_1 = 0$, we have

$$U_{n-1}(1) = \sum_{k=1,3,\ldots}^{n} \binom{n}{k} X_1^{k-1} = \binom{n}{1} = n \,, \tag{82}$$

since, using the fourth line of (81), the only term to survive in the summation is the $k = 1$ term. Similarly, for $X_0 = -1$, we have

$$U_{n-1}(-1) = \sum_{k=1,3,\ldots}^{n} \binom{n}{k} (-1)^{n-k} X_1^{k-1} = (-1)^{n-1} \binom{n}{1} = (-1)^{n-1} n \,. \tag{83}$$

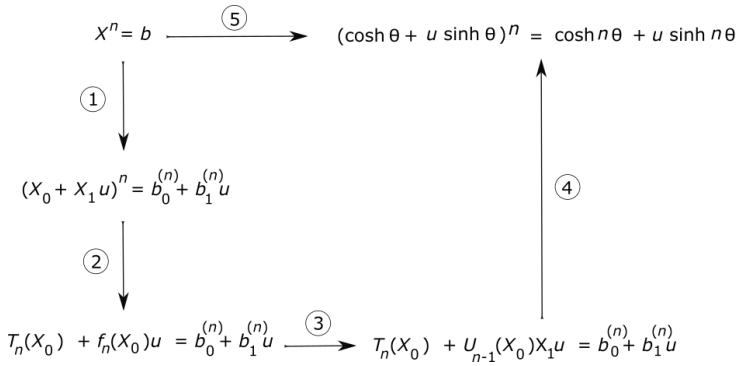

Figure 1: This is how the simple unipodal equation $X^n = b$ with constraint $XX^- = 1$ unites the Chebyshev polynomials of first and second kinds into a single equation and connects their algebra to the circular/hyperbolic functions.

9 The Chebyshev Differential Equation of the Second Kind

Just as (56) led us to (63), the following parallel equation

$$\frac{d^2 b_1^{(n)}}{d\theta^2} - n^2 b_1^{(n)} = 0 \tag{84}$$

will lead to the Chebyshev differential equation of the second kind. To begin with, we solve (72) for $b_1^{(n)}$ to get

$$b_1^{(n)} = X_1 U_{n-1}(X_0) \quad \text{for} \quad n \geq 1\,. \tag{85}$$

On substituting this into (84) with $y = U_{n-1}(X_0)$ for compactness, and noting Eqs. (58) to (61), again with X_0, X_1, and θ real-valued, we have

$$\frac{d^2}{d\theta^2}(X_1 y) - n^2 X_1 y = 0\,, \tag{86}$$

which expands to

$$\big[X_0 \frac{d}{dX_0} + X_1^2 \frac{d^2}{dX_0^2}\big](X_1 y) - n^2 X_1 y = 0\,. \tag{87}$$

After much computation, this becomes

$$(1 - X_0^2)y'' - 3X_0 y' + (n^2 - 1)y = 0\,, \tag{88}$$

where the primes represent derivatives by X_0, and (88) has solution $U_{n-1}(X_0)$. On setting $x = X_0$ and replacing n by $n+1$, we get the familiar differential equation for the Chebyshev polynomials of the second kind, which is satisfied by $U_n(x)$, as

$$(1-x^2)y'' - 3xy' + n(n+2)y = 0\,. \tag{89}$$

10 Unification Update:

Equations (56) and (84) can be unified by the simple unipodal representation

$$\big[\frac{d^2}{d\theta^2} - n^2\big]e^{nu\theta} = 0 \tag{90}$$

where the differential equations for $b_0^{(n)}$ and $b_1^{(n)}$ are contained in, respectively, the complex and uniplex parts of the above equation.

11 Various Lemmas

From here on, we 1) prove many of the relationships (identities) that exist on 1) the Chebyshev polynomials of first and second kinds, mostly employing unipodal methods, 2) unify the Fibonacci and Lucas numbers, and 3) present a unipodal version of the generating functions commonly used for the Chebyshev polynomials.

Lemma 1 *The Chebyshev polynomials have the following properties:*

$$T_n(-x) = (-1)^n T_n(x)\,, \tag{91a}$$
$$U_n(-x) = (-1)^n U_n(x)\,. \tag{91b}$$

Proof: In replacing X_0 by $-X_0$, we induce the transformations:

$$X \to -X^-\,, \qquad X^- \to -X\,. \tag{92}$$

Therefore, applying these transformations to (40), we get

$$\begin{aligned} T_n(-X_0) &= \tfrac{1}{2}[(-X^-)^n + (-X)^n] \\ &= (-1)^n \tfrac{1}{2}[X^{-n} + X^n] \\ &= (-1)^n T_n(X_0)\,. \end{aligned} \tag{93}$$

On replacing X_0 by x, we arrive at (91a).

Now, from (73), we have

$$\begin{aligned} U_n(-X_0) &= \frac{u}{2X_1}[(-X^-)^{n+1} - (-X)^{n+1}] \\ &= \frac{u}{2X_1}(-1)^{n+1}[X^{-(n+1)} - X^{n+1}] \\ &= (-1)^n \frac{u}{2X_1}[X^{n+1} - X^{-(n+1)}] \\ &= (-1)^n U_n(X_0)\,. \end{aligned} \tag{94}$$

On replacing X_0 by x, we arrive at (91b).

Lemma 2 *Given that m and n are positive integers,*

$$2T_mT_n = T_{m+n} + T_{|m-n|}\,, \tag{95a}$$
$$2T_mU_n = U_{m+n} + U_{|m-n|}\,. \tag{95b}$$

Proof: Return to our defining equation (40), we write

$$T_n = \tfrac{1}{2}[X^n + X^{-n}]\,, \tag{96}$$

and thus:

$$\begin{aligned} 2T_mT_n &= \tfrac{1}{2}[X^m + X^{-m}][X^n + X^{-n}] \\ &= \tfrac{1}{2}[X^{m+n} + X^{-(m-n)} + X^{(m-n)} + X^{-(m+n)}] \\ &= \tfrac{1}{2}(X^{m+n} + X^{-(m+n)}) + \tfrac{1}{2}(X^{m-n} + X^{-(m-n)}) \\ &= T_{m+n} + T_{|m-n|}\,. \end{aligned} \tag{97}$$

The proof of (95b) follows similarly.

Lemma 3 *Show that, if $x = \cos\theta$, where θ is a real number, that*

$$U_n(x) = \frac{\sin\,(n+1)\theta}{\sin\theta}\,. \tag{98}$$

Proof: Using (73) gives us

$$U_n(X_0) = \frac{b_1^{(n+1)}}{X_1} \quad \text{for} \quad n \geq 0\,, \tag{99}$$

where $X_0 = \cosh\theta$, $X_1 = \sinh\theta$, and $b_1^{(n+1)} = \sinh\,(n+1)\theta$. Now, replace θ by $i\theta$, where θ is a real number, then $X_0 \to \cos\theta$, $X_1 = \sin\theta$, and $b_1^{(n+1)} = \sin\,(n+1)\theta$, yielding (98).

Lemma 4 *The following square relation between $T_n(x)$ and $U_{n-1}(x)$ holds:*

$$T_n(x)^2 - (x^2-1)U_{n-1}(x)^2 = 1\,. \tag{100}$$

Proof: This result is an immediate consequence of (36), recast in the form

$$b_0^2 - b_1^2 = 1\,. \tag{101}$$

From (37) we have that $b_0 = T_n(X_0)$, and from (72) we have that

$$b_1 = X_1U_{n-1}(X_0)\,. \tag{102}$$

Plugging these values into (101), we get, with x replacing X_0 and $X_1^2 = x^2 - 1$, which yields (100).

Lemma 5 *The following relation holds between $T_n(x)$ and $U_{n-1}(x)$:*

$$T_{n+1}(x) - T_{n-1}(x) = 2(x^2-1)U_{n-1}(x)\,. \tag{103}$$

Proof: We start with

$$U_{n-1}(X_0) = \frac{u}{2X_1}(X^n - X^{-n})\,. \tag{104}$$

Next, we multiply through by $2X_1^2$:

$$2X_1^2 U_{n-1}(X_0) = uX_1(X^n - X^{-n})\,. \tag{105}$$

Now, on the left, replace X_1^2 by $X_0^2 - 1$, and on the right, replace X_1 by $\frac{u}{2}(X - X^{-1})$; and extrapolate:

$$\begin{aligned} 2(X_0^2-1)U_{n-1}(X_0) &= \tfrac{1}{2}(X - X^{-1})(X^n - X^{-n}) \\ &= \tfrac{1}{2}(X^{n+1} + X^{-(n+1)}) - \tfrac{1}{2}(X^{n-1} + X^{-(n-1)}) \\ &= T_{n+1}(X_0) - T_{n-1}(X_0)\,. \end{aligned} \tag{106}$$

Lemma 6 *The following relation holds between $U_{n+1}(x)$, $U_n(x)$, and $U_{n-1}(x)$:*

$$U_n(X_0)^2 - U_{n-1}(X_0)U_{n+1}(X_0) = 1\,. \tag{107}$$

Proof:

$$\begin{aligned} U_n(X_0)^2 - U_{n-1}(X_0)U_{n+1}(X_0) &= \Big(\frac{u}{2X_1}\Big)^2 \Big[(X^{n+1} - X^{-(n+1)})^2 \\ &\qquad - (X^n - X^{-n})[(X^{n+2} - X^{-(n+2)})\Big] \\ &= \frac{1}{4X_1^2}\Big[X^{2n+2} - 2 + X^{-(2n+2)} \\ &\qquad - (X^{2n+2} - X^{-2} - X^2 + X^{-(2n+2)})\Big] \\ &= \frac{1}{4X_1^2}(2T_2(X_0) - 2) \\ &= 1\,. \end{aligned} \tag{108}$$

Lemma 7 *The following is a relation that holds between $U_n(x)$ and $U_m(x)$:*

$$U_{m+n}(X_0) = U_m(X_0)U_n(X_0) - U_{m-1}(X_0)U_{n-1}(X_0)\,. \tag{109}$$

Proof: We begin by rewriting (103), letting $n \to \ell + 1$, as

$$U_\ell(X_0) = \frac{1}{2X_1^2}[T_{\ell+2}(X_0) - T_\ell(X_0)]\,, \tag{110}$$

to be used later. Now,

$$\begin{aligned}
U_m(X_0)U_n(X_0) - U_{m-1}(X_0)U_{n-1}(X_0) = \\
&= \Big(\frac{u}{2X_1}\Big)^2 \Big[(X^{m+1} - X^{-(m+1)})[(X^{n+1} - X^{-(n+1)}) \\
&\qquad\qquad - (X^m - X^{-m})(X^n - X^{-n})\Big] \\
&= \frac{1}{4X_1^2}\Big[X^{m+n+2} - X^{m-n} - X^{-m+n} + X^{-(m+n+2)} \\
&\qquad\qquad - (X^{m+n} - X^{m-n} - X^{-m+n} + X^{-(m+n)})\Big] \\
&= \frac{1}{2X_1^2}[T_{m+n+2}(X_0) - T_{m+n}(X_0)] \\
&= U_{m+n}(X_0)\,, \qquad (111)
\end{aligned}$$

where we invoked (110) with the substitution $\ell \to m + n$.

Lemma 8 *The following is yet another linear relation that holds between $T_{n+1}(x)$ and $U_{n+1}(x)$:*

$$U_{n+1}(x) = xU_n(x) + T_{n+1}(x)\,. \qquad (112)$$

Proof:

From the identity (116) with $n \to n + 1$, we write

$$U_{n+1}(x) = U_{n-1}(x) + 2T_{n+1}(x)\,. \qquad (113)$$

Next, add $U_{n+1}(X_0)$ to both sides and regroup appropriately:

$$2U_{n+1}(x) = [U_{n+1}(x) + U_{n-1}(x)] + 2T_{n+1}(x)\,. \qquad (114)$$

Now, replace $U_{n+1}(x) + U_{n-1}(x)$ by $2xU_n(x)$ and divide through by 2 to get

$$U_{n+1}(x) = xU_n(x) + T_{n+1}(x)\,. \qquad (115)$$

Lemma 9 *The following is a relation that holds between $T_n(x)$ and $U_n(x)$:*

$$2T_\ell(x) = U_\ell(x) - U_{\ell-2}(x)\,. \qquad (116)$$

Proof: We start with

$$T_\ell(X_0) = \tfrac{1}{2}[X^\ell + X^{-\ell}]\,, \qquad (117)$$

and multiply through by

$$1 = \frac{u}{2X_1}[X - X^-]\,, \qquad (118)$$

to get

$$\begin{aligned} T_\ell(X_0) &= \frac{1}{2}\frac{u}{2X_1}[X - X^-][X^\ell + X^{-\ell}] \\ &= \frac{1}{2}\Big[\frac{u}{2X_1}(X^{\ell+1} - X^{-(\ell+1)}) - \frac{u}{2X_1}(X^{\ell-1} - X^{-(\ell-1)})\Big] \\ &= \frac{1}{2}[U_\ell(X_0) - U_{\ell-2}(X_0)]\,, \end{aligned} \tag{119}$$

which is equivalent to (116).

Lemma 10 *The following relation holds (suppressing the functions's arguments):*

$$T_{m+n} = T_m U_n - T_{m-1}U_{n-1}\,. \tag{120}$$

Proof: Using (95b), we can write

$$2T_m U_n = U_{m+n} + U_{|\,m-n\,|}\,, \tag{121a}$$

$$2T_{m-1}U_{n-1} = U_{m+n-2} + U_{|\,m-n\,|}\,. \tag{121b}$$

On subtracting (121b) from (121a), we get

$$2[T_m U_n - T_{m-1}U_{n-1}] = U_{m+n} - U_{m+n-2}\,. \tag{122}$$

Using (116) with $\ell \to m+n$, we have

$$2[T_m U_n - T_{m-1}U_{n-1}] = 2T_{m+n}\,, \tag{123}$$

which is equivalent to (120).

Lemma 11 *Prove the identity for the Chebyshev polynomials*

$$T_{2h+1}^2(x) - x^2 = (x^2-1)(U_{2h}^2(x) - 1)\,, \tag{124}$$

where h is a positive integer.[8]

Proof: Replacing x by X_0, then (124) is equivalent to

$$T_{2h+1}^2(X_0) - 1 = X_1^2 U_{2h}^2(X_0)\,. \tag{125}$$

Let's begin with the RHS and proceed to the LHS:

$$\begin{aligned} X_1^2 U_{2h}^2(X_0) &= X_1^2\big[\frac{u}{2X_1}(X^{2h+1} - X^{-(2h+1)})\big]^2 \\ &= \frac{1}{4}(X^{2(2h+1)} - 2 + X^{-2(2h+1)}) \\ &= \frac{1}{4}(X^{2(2h+1)} + 2 + X^{-2(2h+1)}) - 1 \\ &= T_{2h+1}^2(X_0) - 1\,. \end{aligned} \tag{126}$$

[8] I found this identity on page 4 of the preprint paper [7].

Lemma 12 *The following relations between $T_n(x)$ and $U_{n-1}(x)$ hold*

$$T_n(x) + U_{n-1}(x)x_1 = x_+^n \,. \tag{127a}$$
$$T_n(x) - U_{n-1}(x)x_1 = x_-^n \,, \tag{127b}$$

where $x_1 = \sqrt{x^2-1}$.

Proof: From (28) we can write

$$b_0^{(n)} + b_1^{(n)} = X_+^n \,, \tag{128a}$$
$$b_0^{(n)} - b_1^{(n)} = X_-^n \,. \tag{128b}$$

But $b_0^{(n)} = T_n(X_0)$ and $b_1^{(n)} = U_{n-1}(X_0)X_1$, therefore

$$T_n(X_0) + X_1 U_{n-1}(X_0) = X_+^n \,. \tag{129a}$$
$$T_n(X_0) - X_1 U_{n-1}(X_0) = X_-^n \,. \tag{129b}$$

To arrive at the given relations, we need only replace X_0 by x.

Lemma 13 *Proof of the composition identity:*

$$U_n(T_m(x)) = \frac{U_{m(n+1)-1}(x)}{U_{m-1}(x)} \,. \tag{130}$$

Proof: With the help of (77b), we begin with

$$U_n(Y_0) = \frac{1}{2Y_1}(Y_+^{n+1} - Y_-^{n+1}) \,, \tag{131}$$

where Y_0 is a temporary variable, to be replaced by $T_m(X_0)$, and, of course,

$$Y_0^2 - Y_1^2 = 1 \,. \tag{132}$$

From a modified version (100), we have that

$$T_m(X_0)^2 - 1 = X_1^2 U_{m-1}(X_0)^2 \,. \tag{133}$$

On replacing Y_0 by $T_m(X_0)$, we get

$$Y_1 = \sqrt{Y_0^2 - 1} = \sqrt{T_m(X_0)^2 - 1} = X_1 U_{m-1}(X_0) \,. \tag{134}$$

Now that we have Y_0 and Y_1, using (129a) and (129b), we can write

$$Y_+ = Y_0 + Y_1 = T_m(X_0) + X_1 U_{m-1}(X_0) = X_+^m \,, \tag{135a}$$
$$Y_- = Y_0 - Y_1 = T_m(X_0) - X_1 U_{m-1}(X_0) = X_-^m \,. \tag{135b}$$

Thus,

$$\begin{aligned} U_n(T_m(X_0)) &= \frac{1}{2X_1U_{m-1}(X_0)}\big[(X_+^m)^{n+1} - (X_-^m)^{n+1}\big] \\ &= \frac{1}{U_{m-1}(X_0)}\left[\frac{1}{2X_1}\big(X_+^{m(n+1)} - X_-^{m(n+1)}\big)\right] \\ &= \frac{U_{m(n+1)-1}(X_0)}{U_{m-1}(X_0)}\,. \end{aligned} \tag{136}$$

On page 2159 of [2] is found a proof of a similar version of the identity.

Lemma 14 *Given m and n as positive integers, then*

$$T_n(T_m(X_0)) = T_{nm}(X_0)\,. \tag{137}$$

Proof: Starting with (48) and using the machinary of the above lemma, we write

$$T_n(Y_0) = \tfrac{1}{2}(Y_+^n + Y_-^n)\,, \tag{138}$$

where, as in the last lemma,

$$Y_+ = X_+^m \quad \text{and} \quad Y_- = X_-^m\,. \tag{139}$$

On substituting these values into (138), we get

$$T_n(T_m(X_0)) = \tfrac{1}{2}(X_+^{nm} + X_-^{nm}) = T_{nm}(X_0)\,. \tag{140}$$

Lemma 15 *For an expression for the Chebyshev polynomials of the first kind in X_0, we have*

$$T_n(X_0) = \sum_{\ell=0,1,\ldots}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2\ell} X_0^{n-2\ell}(X_0^2-1)^\ell\,. \tag{141}$$

Before we begin, given that $T_n(X_0)$ is pure complex (has no unisor part), it's good to anticipate that the uniplex part of the expansion of (40) is identically zero. Also, note that for any nonnegative integer k:

$$[1+(-1)^k]u^k = \begin{cases} 2\,, & \text{for } k \text{ even}, \\ 0\,, & \text{for } k \text{ odd}. \end{cases} \tag{142}$$

Note: We will be using the floor function $\lfloor\cdot\rfloor$, which operates on real numbers and returns the largest integer less than or equal to its argument.

Proof: Now, we expand both X^n and $X^{-n} = (X^-)^n$ with the binomial formula.

$$\begin{aligned} T_n(X_0) &= \tfrac{1}{2}[X^n + (X^-)^n] \\ &= \tfrac{1}{2}[(X_0 + X_1 u)^n + (X_0 - X_1 u)^n] \\ &= \tfrac{1}{2}\Big[\sum_{k=0}^{n} \binom{n}{k} X_0^{n-k} X_1^k [1 + (-1)^k] u^k\Big] \\ &= \sum_{k=0,2,\ldots}^{n} \binom{n}{k} X_0^{n-k} X_1^k \\ &= \sum_{\ell=0,1,\ldots}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2\ell} X_0^{n-2\ell} X_1^{2\ell} \\ &= \sum_{\ell=0,1,\ldots}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2\ell} X_0^{n-2\ell} (X_0^2 - 1)^\ell \,. \end{aligned}$$

Corollary

Since $T_n(x) = b_0^{(n)} = \cosh n\theta$, then we have from (141)

$$\cosh n\theta = \sum_{\ell=0,1,\ldots}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2\ell} (\cosh^{n-2\ell}\theta)(\cosh^2\theta - 1)^\ell \,. \tag{143}$$

On replacing θ by $i\theta$ in this last equation, we get

$$\cos n\theta = \sum_{\ell=0,1,\ldots}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2\ell} (\cos^{n-2\ell}\theta)(\cos^2\theta - 1)^\ell \,. \tag{144}$$

Lemma 16 *The Chebyshev polynomials of the first kind have first derivative:*

$$\frac{dT_n(x)}{dx} = nU_{n-1}(x) \,. \tag{145}$$

Proof: Before we take the derivative of (40) by X_0, we need some preliminary results. Taking both X_0 and X_1 as real-valued, by implicit differentiation of $X_0^2 - X_1^2 = 1$ by X_0, we get

$$\frac{dX_1}{dX_0} = \frac{X_0}{X_1} \,. \tag{146}$$

Therefore

$$\frac{dX}{dX_0} = \frac{u}{X_1} X \quad \text{and} \quad \frac{dX^-}{dX_0} = -\frac{u}{X_1} X^- \,, \tag{147}$$

and thus,

$$\begin{aligned}\frac{dT_n(X_0)}{dX_0} &= \frac{d}{dX_0}\frac{1}{2}[X^n + (X^-)^n] \\ &= \frac{nu}{2X_1}[X^n - (X^-)^n] \\ &= nU_{n-1}(X_0)\,. \end{aligned} \tag{148}$$

Now replace X_0 by x.

Lemma 17 *The Chebyshev polynomials of the first kind have the first integrals:*

$$\int T_n(x)\,dx = \frac{1}{2}\Big[\frac{T_{n+1}(x)}{n+1} - \frac{T_{n-1}(x)}{n-1}\Big] + C\,, \quad n \geq 2\,, \tag{149}$$

where C is an arbitrary constant. (Integrals for T_0 and T_1 are trivial.)

Proof #1: This time we use hyperbolic functions, but with variables X_0, X_1, and θ real-valued. Since $T_n(X_0) = \cosh n\theta$ and $dX_0 = \sinh\theta\,d\theta$, then

$$\begin{aligned}\int T_n(X_0)\,dX_0 &= \int \frac{1}{2}(e^{un\theta} + e^{-un\theta})\frac{1}{2u}(e^{u\theta} - e^{-u\theta})\,d\theta \\ &= \frac{1}{4u}\int e^{u(n+1)\theta} - e^{u(n-1)\theta} + e^{-u(n-1)\theta} - e^{-u(n+1)\theta})\,d\theta \\ &= \frac{1}{4}\Big[\frac{e^{u(n+1)\theta} + e^{-u(n+1)\theta}}{n+1} - \frac{e^{u(n-1)\theta} + e^{-u(n-1)\theta}}{n-1}\Big] + C \\ &= \frac{1}{2}\Big[\frac{\cosh\,(n+1)\theta}{n+1} - \frac{\cosh\,(n-1)\theta}{n-1}\Big] + C \\ &= \frac{1}{2}\Big[\frac{T_{n+1}(X_0)}{n+1} - \frac{T_{n-1}(X_0)}{n-1}\Big] + C\,. \end{aligned} \tag{150}$$

Proof #2: This time we fulfil the promise of not needing trig/hyperbolic functions to prove identities (with variables X_0 and X_1 still real-valued). Since,

$$X_1 = \frac{u}{2}(X - X^-)\,, \tag{151}$$

and, from (147), we have that

$$\frac{dX_0}{dX} = uX_1X^- = \frac{1}{2}(X - X^-)X^- = \frac{1}{2}(1 - X^{-2})\,, \tag{152}$$

and, since $dX_0 = \dfrac{dX_0}{dX} dX$, then

$$\begin{aligned}
\int T_n(X_0)\, dX_0 &= \int \frac{1}{2}(X^n + X^{-n}) \frac{dX_0}{dX} dX \\
&= \int \frac{1}{2}(X^n + X^{-n}) \frac{1}{2}(1 - X^{-2}) dX \\
&= \frac{1}{2}\Big[\frac{1}{2}\frac{X^{n+1} + X^{-(n+1)}}{n+1} - \frac{1}{2}\frac{X^{n-1} + X^{-(n-1)}}{n-1}\Big] + C \\
&= \frac{1}{2}\Big[\frac{T_{n+1}(X_0)}{n+1} - \frac{T_{n-1}(X_0)}{n-1}\Big] + C\,. \qquad (153)
\end{aligned}$$

Lemma 18 *The Chebyshev polynomials of the second kind have first derivatives:*

$$\frac{dU_n(x)}{dx} = \frac{(n+1)T_{n+1}(x) - xU_n(x)}{x^2 - 1}\,. \qquad (154)$$

Proof: We differentiate

$$U_{n-1}(X_0) = \frac{u}{2X_1}[X^n - X^{-n}]\,, \qquad (155)$$

by X_0 (real-valued), using the product rule, and with the help of (146) and (147), we get

$$\begin{aligned}
\frac{dU_{n-1}(X_0)}{dX_0} &= \frac{d}{dX_0}\frac{u}{2X_1}[X^n - X^{-n}] \\
&= -\frac{u}{2X_1^2}\frac{X_0}{X_1}[X^n - X^{-n}] + \frac{u}{2X_1}[nX^{n-1} - (-n)X^{-n-1}]\frac{uX}{X_1} \\
&= -\frac{X_0}{X_1^2}U_{n-1}(X_0) + \frac{n}{2X_1^2}[X^n + X^{-n}] \\
&= \frac{nT_n(X_0) - X_0 U_{n-1}(X_0)}{X_0^2 - 1}\,. \qquad (156)
\end{aligned}$$

Now replace X_0 by x and let n go to $n+1$ to derive (154).

12 Chebyshev Relation to Fibonacci and Lucas Numbers

The Lucas and Fibonacci numbers are, respectively, defined by the recurrence relations:

$$L_{n+2} = L_{n+1} + L_n \quad \text{where} \quad L_0 = 2,\ L_1 = 1\,, \qquad (157a)$$

$$F_{n+2} = F_{n+1} + F_n \quad \text{where} \quad F_0 = 0,\ F_1 = 1\,. \qquad (157b)$$

Using the unipodal algebra, prove that the well-known connections of the Chebyshev polynomials to the Fibonacci numbers F_n and the Lucas numbers L_n are, respectively,

$$F_n = i^{-n+1} U_{n-1}(i/2) \quad \text{for} \quad n \geq 0\,, \tag{158a}$$

$$L_n = 2i^{-n} T_n(i/2) \quad \text{for} \quad n \geq 0\,. \tag{158b}$$

See [3] for a reference.
Proof: First, a bit of notation from Fibonacci theory. Let

$$\varphi_+ \equiv \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \varphi_- \equiv \frac{1-\sqrt{5}}{2}\,. \tag{159}$$

Then, the nth Fibonacci number can, remarkably, be given as

$$F_n = \frac{\varphi_+^n - \varphi_-^n}{\sqrt{5}}\,, \tag{160}$$

which is known as the *Binet Formula*.

Using the alternative equation for U_{n-1} from (77b), we have,

$$U_{n-1}(X_0) = \frac{X_+^n - X_-^n}{2X_1}\,. \tag{161}$$

Now, if we set $X_0 = i/2$ then $X_1 = i\sqrt{5}/2$, $X_+ = i\varphi_+$, and $X_- = i\varphi_-$

$$U_{n-1}(i/2) = i^n \frac{\varphi_+^n - \varphi_-^n}{i\sqrt{5}}\,. \tag{162}$$

Comparing this last equation to (160), we get

$$F_n = i^{-n+1} U_{n-1}(i/2) = \frac{U_{n-1}(i/2)}{i^{n-1}}\,, \tag{163}$$

where $U_{-1} = 0$ is consistent with $f_0 = 0$ in (70).

Proving the relation in (158b) for the Lucas numbers is even easier. Employing the literature, we accept that

$$L_n = \varphi_+^n + \varphi_-^n\,. \tag{164}$$

With the help of (48) we can write

$$\begin{aligned} 2i^{-n} T_n(i/2) &= i^{-n}(X_+^n + X_-^n)|_{X_0 = i/2} \\ &= \varphi_+^n + \varphi_-^n \\ &= L_n\,. \end{aligned} \tag{165}$$

Reader Exercise: Prove *Cassini's Identity:*

$$F_{n-1} F_{n+1} - F_n^2 = (-1)^n\,, \tag{166}$$

using (107) and that $U_n(i/2) = i^n F_{n+1}$.

Reader Exercise: Prove the identity:

$$L_{m+n} = L_{m+1}F_n + L_m F_{n-1}\,, \tag{167}$$

using (120) and that $L_{m+n} = \dfrac{2}{i^{m+n}} T_{m+n}(i/2)$, adapted from (158b). Hint: Let $\overline{T}_n = T_n(i/2)$ and $\overline{U}_{n-1} = U_{n-1}(i/2)$ and rewrite Equations (158a) and (158b) as

$$\overline{U}_{n-1} = i^{-n+1} F_n \quad \text{for} \quad n \geq 0\,, \tag{168a}$$

$$\overline{T}_n = 2i^{-n} L_n \quad \text{for} \quad n \geq 0\,, \tag{168b}$$

and substitute into $L_{m+n} = \dfrac{2}{i^{m+n}} \overline{T}_{m+n}$. The first result is

$$L_{m+n} = L_m F_{n+1} + L_{m-1} F_n\,. \tag{169}$$

Now, replace m by $m+1$ and n by $n-1$.

13 First Unipodal Unification of Fibonacci and Lucas Numbers

There are a number of ways of unifying the Lucas and Fibonacci numbers within the unipodal algebra. Using the combined expressions given in (158a) and (158b) is one way. We can make them more explicit, thusly,

$$F_n = \frac{u}{i^n \sqrt{5}} (\bar{X}^n - \bar{X}^{-n})\,, \tag{170}$$

$$L_n = \frac{1}{i^n} (\bar{X}^n + \bar{X}^{-n})\,, \tag{171}$$

where $\bar{X} \equiv \bar{X}_0 + \bar{X}_1 u = \dfrac{i}{2} + \dfrac{i\sqrt{5}\,u}{2}$, and, of course, $\bar{X}\bar{X}^- = 1$.

Reader Exercises: Use (170) and (171) to show that

$$F_n L_m = F_{n+m} + (-1)^m F_{n-m}\,, \tag{172}$$

$$L_{2n} = L_n^2 - 2(-1)^n\,. \tag{173}$$

Reader Exercise: Prove the generalization of Cassini's Identity, know as the *Catalan's Identity:*

$$F_n^2 - F_{n-r} F_{n+r} = (-1)^{n-r} F_r^2\,. \tag{174}$$

Hint: Use Equation (170) to show that $F_{n-r}F_{n+r}$ and $F_n^2 - (-1)^{n-r}F_r^2$ have the same representation in terms of $\bar{X}$.

Since T_n and U_n are defined on negative numbers, we can extend F_n and L_n to negative numbers, as well.

$$F_{-n} = (-1)^{n+1} F_n \,, \tag{175}$$

$$L_{-n} = (-1)^n L_n \,. \tag{176}$$

Reader Exercise: Show that

$$F_n = \frac{1}{i^n \sqrt{5}} (\bar{X}_+^n - \bar{X}_-^n) \,, \tag{177}$$

$$L_n = \frac{1}{i^n} (\bar{X}_+^n + \bar{X}_-^n) \,. \tag{178}$$

Reader Exercise: Using (177) and (178), show on one line that $F_n L_n = F_{2n}$.

14 Second Unipodal Unification of Fibonacci and Lucas Numbers

Another way to unify the Fibonacci and Lucas numbers begins with the following definition:

$$\Phi \equiv \varphi_+ u_+ + \varphi_- u_- \,, \tag{179}$$

or, in the standard basis,

$$\Phi = \tfrac{1}{2}(1 + \sqrt{5} u) = -i \bar{X} \,. \tag{180}$$

Adding in the identities,

$$\varphi_+ + \varphi_- = 1 \,, \qquad \varphi_+ - \varphi_- = \sqrt{5} \,, \tag{181a}$$

$$\varphi_+^{-1} = -\varphi_- \,, \qquad \varphi_-^{-1} = -\varphi_+ \,, \tag{181b}$$

$$\varphi_+ \varphi_- = -1 \,, \tag{181c}$$

together with the additional identities,

$$\Phi^{-1} = -\varphi_- u_+ - \varphi_+ u_- \,, \tag{182a}$$

$$\Phi^- = \varphi_+ u_- + \varphi_- u_+ \,, \tag{182b}$$

$$\Phi \Phi^- = -1 \,. \tag{182c}$$

$$\Phi^- = -\Phi^{-1} \,, \tag{182d}$$

$$\Phi \Phi^- = -1 \,, \tag{182e}$$

$$\Phi + \Phi^- = 1 \,, \tag{182f}$$

$$\Phi - \Phi^- = \sqrt{5} u \,, \tag{182g}$$

$$\Phi^2 = \tfrac{1}{2}(3 + \sqrt{5} u) \,, \tag{182h}$$

$$\Phi^2 - \Phi - 1 = 0 \,, \tag{182i}$$

we can now produce a few of the standard Fibonacci-Lucas mixed identities. Now,

$$\Phi^n = \varphi_+^n u_+ + \varphi_-^n u_- \quad \text{(Idempotent Basis)}\,, \tag{183a}$$

$$\Phi^n = \frac{1}{2}L_n + \frac{\sqrt{5}}{2}F_n u \quad \text{(Standard Basis)}\,, \tag{183b}$$

where we used (160) and (164).

From (182e), we have

$$(-1)^n = (\Phi\Phi^-)^n = \Phi^n(\Phi^n)^-\,, \tag{184}$$

from which with (183b) the identity follows:

$$L_n^2 - 5F_n^2 = 4(-1)^n\,. \tag{185}$$

Since $\varphi_+ + \varphi_- = 1$, then, using (181c), we get

$$\begin{aligned} \Phi^n &= (\varphi_+ + \varphi_-)\Phi^n \\ \varphi_+^n u_+ + \varphi_-^n u_- &= (\varphi_+^{n+1} - \varphi_+^{n-1})u_+ + (\varphi_-^{n+1} - \varphi_-^{n-1})u_-\,. \end{aligned} \tag{186a}$$

We get two important results from this. First, equate, respectively, the u_+ and u_- parts, to get

$$\varphi_+^n = \varphi_+^{n+1} - \varphi_+^{n-1}\,, \tag{187a}$$

$$\varphi_-^n = \varphi_-^{n+1} - \varphi_-^{n-1}\,. \tag{187b}$$

Second, convert (186a) to standard basis and use (160) and (164) to get functions of L and F. Then equate, respectively, the complex and uniplex parts, to get

$$L_n = L_{n+1} - L_{n-1}\,, \tag{188a}$$

$$F_n = F_{n+1} - F_{n-1}\,. \tag{188b}$$

At this point, I leave the further exploration of this unipodal unification of Fibonacci-Lucas numbers to someone else.

15 The Unipodal Generating Function for the Chebyshev Polynomials

Although we have accomplished all our proofs concerning the Chebyshev polynomials thus far without the need of their generating functions, we will at this time produce those functions by use of the unipodal algebra. The relevant literature on the subject tells us that the g.f. for the Chebyshev polynomials of the first kind is

$$\frac{1 - tx}{1 - 2tx + t^2} = \sum_{n=0}^{\infty} T_n(x)t^n\,, \tag{189}$$

and for the second kind is

$$\frac{1}{1-2tx+t^2} = \sum_{n=0}^{\infty} U_n(x)t^n \,. \tag{190}$$

We can enfold both of these relations into the single unipodal equation:

$$\frac{1}{1-te^{u\theta}} = \sum_{n=0}^{\infty} (te^{u\theta})^n = \sum_{n=0}^{\infty} t^n b^{(n)} \,. \tag{191}$$

Proof: First, (191) is true because a) of the theorem on geometric series and b) because of (34). Note,

$$e^{u\theta} = X_0 + X_1 u \,, \quad \text{and} \quad e^{-u\theta} = X_0 - X_1 u \,. \tag{192}$$

Next, on complexifying the denominator of the LHS of (191), we get the new unipodal equation

$$\frac{1-te^{-u\theta}}{1-2tX_0+t^2} = \sum_{n=0}^{\infty} b^{(n)} t^n \,. \tag{193}$$

On equating the complex parts, we get

$$\frac{1-tX_0}{1-2tX_0+t^2} = \sum_{n=0}^{\infty} b_0^{(n)} t^n \,. \tag{194}$$

Replacing $b_0^{(n)}$ by $T_n(X_0)$, yields (189). On equating the uniplex parts of (193) and using (85), we get

$$\frac{tX_1}{1-2tX_0+t^2} = \sum_{n=0}^{\infty} b_1^{(n)} t^n = \sum_{n=0}^{\infty} X_1 U_{n-1}(X_0) t^n \,. \tag{195}$$

However, this lower limit on the right of $n = 0$ will make more sense after we divide through by tX_1, yielding:

$$\frac{1}{1-2tX_0+t^2} = \sum_{n=1}^{\infty} U_{n-1}(X_0) t^{n-1} \,, \tag{196}$$

which we can rewrite as

$$\frac{1}{1-2tX_0+t^2} = \sum_{n=0}^{\infty} U_n(X_0) t^n \,, \tag{197}$$

which is (190). I don't know how to best make use of (191). Perhaps it can be used to reformulate the Gegenbauer polynomials into the unipodal algebra to some benefit.

16 Conclusion

The unipodal algebra is a commutative algebra that sits between the complex algebra on the low end and the quaternions and higher-dimensional Clifford algebras on the high end. By its use, we have been able to unify the treatment of the Chebyshev polynomials of first and second kinds more elegantly than can be done by use of complex numbers alone, principally because, by its access to an idempotent basis, we have the superior means to generate the standard identities. We have also been able to unify the Lucas and Fibonacci numbers in two different ways.

We have shown that the unipodal algebra allows for a compact unification of the Chebyshev polynomials of the first and second kinds into a single unipodal generating function, but I have yet to find any use of this unipodal generating function that is more elegant in proofs than the proofs shown above.

Without forfeiting any capability for use of circular/hyperbolic functions in proofs of identities, the unipodal algebra allows us alternatively to use more 'algebraic' proofs, by using relatively simple polynomials in X and X^{-}.

Appendix 1:

a) First conventional proof of the formula for the ordinary generating function for the Chebyshev polynomials of the first kind:

$$\sum_{n=0}^{\infty} T_n(x)t^n = \frac{1-tx}{1-2tx+t^2}\,. \tag{198}$$

We begin by setting $S = \sum_{n=0}^{\infty} T_n(x)t^n$. Now, it stands to reason that we will need to incorporate (43) in our proof somewhere, so let's get right to it. (We'll replace X_0 by x.) With $T_0 = 1$ and $T_1 = x$, we have

$$\begin{aligned} S &= 1 + xt + \sum_{n=2}^{\infty} T_n t^n \\ &= 1 + xt + \sum_{n=0}^{\infty} T_{n+2} t^{n+2} \\ &= 1 + xt + \sum_{n=0}^{\infty} [2xT_{n+1} - T_n] t^{n+2} \\ &= 1 + xt + 2xt \sum_{n=0}^{\infty} T_{n+1} t^{n+1} - t^2 \sum_{n=0}^{\infty} T_n t^n \\ &= 1 + xt + 2xt(S-1) - t^2 S\,. \end{aligned} \tag{199}$$

On solving this last equation for S, we have

$$S = \frac{1-tx}{1-2tx+t^2}\,, \tag{200}$$

which is what we needed to show.

b) Second conventional proof of the formula: This time we start with the identity over the complex numbers

$$\frac{1}{1-te^{i\theta}} = \sum_{n=0}^{\infty} (te^{i\theta})^n\,, \tag{201}$$

and equate the real parts of both sides, setting $\cos\theta = x$:

$$\frac{1-tx}{1-2tx+t^2} = \sum_{n=0}^{\infty} \cos\,(n\theta)\, t^n\,. \tag{202}$$

But what is $\cos\,(n\theta)$? It's $b_0^{(n)}$. Or, rather I should say that it's $b_0^{(n)}$ when θ is replaced by a pure imaginary angle $i\theta$.

$$\frac{1-tx}{1-2tx+t^2} = \sum_{n=0}^{\infty} b_0^{(n)}\, t^n\,. \tag{203}$$

But, according to (37), $T_n(X_0) = b_0^{(n)}$. Thus, substituting this into the last equation yields (198).

As a comment, I must say that this seems nothing less than extraordinary. In the first proof we seeded the expansion with $T_0 = 1$ and $T_1 = x$. In the second proof, we didn't even need that. It's as if the Chebyshev polynomials are the most natural and simple functions that form a bridge between ordinary algebra and trigonometric/hyperbolic functions.

Appendix 2

Here, we'll use the generating function for the Chebyshev polynomials of the second kind to solve for recurrence relation the U_n obey, namely, Eq. (76). We start with (197) and then differentiate it by t, to get

$$\frac{-2x + 2t}{(1 - 2tx + t^2)^2} = \sum_{n=0}^{\infty} nU_n(x)t^{n-1} \,. \tag{204}$$

Next, we multiply through by $1 - 2tx + t^2$ and use (197) on the LHS, to get

$$(-2x + 2t) \sum_{n=0}^{\infty} U_n(x)t^n = (1 - 2tx + t^2) \sum_{n=0}^{\infty} nU_n(x)t^{n-1} \,. \tag{205}$$

Putting the nonzero terms on the RHS, and simplifying, we get

$$0 = \sum_{n=0}^{\infty} nU_n(x)t^{n-1} - 2x \sum_{n=0}^{\infty} (n+1)U_n(x)t^n + \sum_{n=0}^{\infty} (n+2)U_n(x)t^{n+1} \,. \tag{206}$$

For the first term on the RHS, we let

$$\sum_{n=0}^{\infty} nU_n(x)t^{n-1} \to \sum_{n=0}^{\infty} (n+1)U_{n+1}(x)t^n \,. \tag{207}$$

For the third term, we let

$$\sum_{n=0}^{\infty} (n+2)U_n(x)t^{n+1} \to \sum_{n=0}^{\infty} (n+1)U_{n-1}(x)t^n \,, \tag{208}$$

where we used that $U_{-1} = 0$, which was proved on page 13. Putting this altogether in (206), yields

$$0 = \sum_{n=0}^{\infty} (n+1)t^n \big[U_{n+1}(x) - 2xU_n(x) + U_{n-1}(x)\big] \,. \tag{209}$$

Setting each term equal to zero for each n, yields (76).

Appendix 3

What follows is the author's (X. Li's) proof an identity for the Chebyshev polynomials that appeared on page 4 of [7]:

$$\begin{aligned} T_{2h+1}^2(x) - x^2 &= (xU_{2h}(x) - U_{2h-1}(x))^2 - x^2 \\ &= x^2U_{2h}^2(x) - 2xU_{2h}(x)U_{2h-1}(x) + U_{2h-1}^2(x) - x^2 \\ &= x^2(U_{2h}^2(x) - 1) - U_{2h-1}(x)U_{2h+1}(x) \\ &= x^2(U_{2h}^2(x) - 1) - \frac{1}{4(x^2-1)} \cdot \left[(\alpha^{2h+2} - \beta^{2h+2})(\alpha^{2h} - (\beta^{2h})\right] \\ &= x^2(U_{2h}^2(x) - 1) - \frac{1}{4(x^2-1)} \cdot \left[(\alpha^{2h+1} - \beta^{2h+1})^2 - (\alpha - \beta)^2\right] \\ &= x^2(U_{2h}^2(x) - 1) - U_{2h}^2(x) + U_0^2(x) \\ &= (x^2-1)(U_{2h}^2(x) - 1) \end{aligned} \tag{210}$$

Done. The unipodal proof is found on page 20 of this paper.

Appendix 4

The $T_m(U_n(x))$ composition identity:

$$T_m(U_n(x)) = \sum_{\ell=0,1,\ldots}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2\ell} U_n(X_0)^{m-2\ell} U_{n+1}(X_0)^\ell U_{n-1}(X_0)^\ell \,. \tag{211}$$

Proof: If you remember from Lemma 18, the 'first' composition identity was the simple:

$$U_n(T_m(x)) = \frac{U_{m(n+1)-1}(x)}{U_{m-1}(x)} \,. \tag{212}$$

However, I found within the literature no comparably simple form for $T_m(U_n(x))$. If we proceed as we did before, with the help of (48), we begin with

$$T_m(Y_0) = \frac{1}{2}(Y_+^m + Y_-^m) \,, \tag{213}$$

where we anticipate that Y_0 will be replaced by $U_n(X_0)$, which implies that

$$Y_1 = \sqrt{U_n(X_0)^2 - 1} \,. \tag{214}$$

Therefore,

$$Y_+ = Y_0 + Y_1 = U_n(X_0) + \sqrt{U_n(X_0)^2 - 1} \,, \tag{215a}$$

$$Y_- = Y_0 - Y_1 = U_n(X_0) - \sqrt{U_n(X_0)^2 - 1} \,, \tag{215b}$$

but where do we go from here? We may as well just use the formula from (141). Thus, with the help of (107), we get

$$T_m(Y_0) = \sum_{\ell=0,1,\ldots}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2\ell} Y_0^{m-2\ell} (Y_0^2 - 1)^\ell \,, \tag{216a}$$

$$T_m(U_n(X_0)) = \sum_{\ell=0,1,\ldots}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2\ell} U_n(X_0)^{m-2\ell} (U_n(X_0)^2 - 1)^\ell \,, \tag{216b}$$

$$T_m(U_n(X_0)) = \sum_{\ell=0,1,\ldots}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2\ell} U_n(X_0)^{m-2\ell} U_{n+1}(X_0)^\ell U_{n-1}(X_0)^\ell \,, \tag{216c}$$

if that leaves us any the wiser.

Corollary:

$$T_m(2X_0) = \sum_{\ell=0,1,\ldots}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2\ell} (2X_0)^{m-2\ell} (4X_0^2 - 1)^\ell \,, \tag{217}$$

where we used that $T_m(2X_0)) = T_m(U_1(X_0))$.

Appendix 5

The $U_m(U_n(x))$ composition identity:

$$U_m(U_n(x)) = \sum_{\ell=1,\ldots}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2\ell} U_n(X_0)^{m-2\ell+1} (U_n(X_0)^2 - 1)^{\ell-1} \,. \tag{218}$$

Proof: If we proceed as we did before, with the help of (77b), we begin with

$$U_m(Y_0) = \frac{1}{2}(Y_+^m + Y_-^m) \,, \tag{219}$$

where we anticipate that Y_0 will be replaced by $U_n(X_0)$. Then, to have that

$$Y_0^2 - Y_1^2 = 1 \,, \tag{220}$$

we must have that

$$Y_1 = \sqrt{U_n(X_0)^2 - 1} \,, \tag{221}$$

where we have used a modified version of (100). Therefore,

$$Y_+ = Y_0 + Y_1 = U_n(X_0) + \sqrt{U_n(X_0)^2 - 1} \,, \tag{222a}$$

$$Y_- = Y_0 - Y_1 = U_n(X_0) - \sqrt{U_n(X_0)^2 - 1} \,, \tag{222b}$$

but where do we go from here? We may as well just use the formula from (79). Thus, we get

$$U_m(Y_0) = \sum_{\ell=0,1,\ldots}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2\ell-1} Y_0^{n-2\ell+1} (Y_0^2-1)^{\ell-1}\,, \tag{223a}$$

$$U_m(U_n(X_0)) = \sum_{\ell=1,\ldots}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2\ell-1} U_n(X_0)^{m-2\ell+1} (U_n(X_0)^2-1)^{\ell-1}\,, \tag{223b}$$

if that leaves us any the wiser.

Corollary:

$$U_m(2X_0) = \sum_{\ell=0,1,\ldots}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2\ell} (2X_0)^{m-2\ell} (4X_0^2-1)^{\ell}\,, \tag{224}$$

where we used that $U_m(2X_0)) = U_m(U_1(X_0))$.

Appendix 6

I include the following identity, not because I've found a unipodal proof of it, but because it's just interesting to me.

$$U_n(X_0) = \sum_{j=0}^{n} X_0^j T_{n-j}(X_0)\,. \tag{225}$$

We will use induction to prove this equality. First, we test the base case: $U_0 = 1 = X_0^0 T_0$. Check! Next, our inductive hypothesis is that (225) is true for some particular n and we must show that it is true for $n+1$, as well.

$$\begin{aligned}
\sum_{j=0}^{n+1} X_0^j T_{n+1-j}(X_0) &= T_{n+1} + X_0 T_n + X_0^2 T_{n-1} + \ldots + X_0^{n+1} \\
&= T_{n+1} + X_0 \big[T_n + X_0 T_{n-1} + \ldots + X_0^n \big] \\
&= T_{n+1} + X_0 \sum_{j=0}^{n} X_0^j T_{n-j}(X_0) \\
&= T_{n+1} + X_0 U_n(X_0)\,,
\end{aligned} \tag{226}$$

where we used the inductive hypothesis to get the last step. But is $T_{n+1} + X_0 U_n(X_0)$ equal to $U_{n+1}(X_0)$? From (112) we know that it is, and the proof is finished.

Appendix 7

The following polynomial equation turned up in my 1994 paper, though I used $f_n(X_0)$ instead of $T_n(X_0)$:

$$X^{2n} - 2T_n(X_0)X^n + 1 = 0\,, \tag{227}$$

where $T_n = b_0$ is representation of $\cosh n\theta$ in terms of a polynomial in $\cosh\theta$:

$$T_n(\cosh\theta) = \cosh n\theta\,. \tag{228}$$

(See above Equations (37) and (38).) I played around with (227) and ended up with

$$T_n(X_0) = 2X_0T_{n-1}(X_0) - T_{n-2}(X_0)\,, \quad \text{for } n \geq 2\,, \tag{229}$$

though I didn't recognize it at that time as the recurrence relation for the Chebyshev polynomials of the first kind. I include the following inferior proof (to that presented above in Section 5) for historical purposes, as well as for connection to the work of Roger Cotes (1682–1716), as presented in the book *Galois' theory of Algebraic Equations* by Jean-Pierre Tignol [13].

In the calculations that follow, we will suppress the argument of T_n. Thus, we have that

$$X^{2n} + 1 = 2T_nX^n\,, \tag{230}$$

Now, on replacing n by $n-1$ in (230), gives

$$X^{2n-2} + 1 = 2T_{n-1}X^{n-1}\,. \tag{231}$$

To find a recurrence relation on the T's, our goal is to remove any term that does not have a T in it. Our first step toward this goal is to subtract (230) from (231) to eliminate the constant term, yielding,

$$X^{2n-2} - X^{2n} = 2T_{n-1}X^{n-1} - 2T_nX^n\,. \tag{232}$$

On dividing this last equation by X^n, we get

$$X^{n-2} - X^n = 2T_{n-1}X^{-1} - 2T_n\,. \tag{233}$$

Now, replacing n by $n-1$ in this last equation and then multiplying the result by X gives

$$X^{n-2} - X^n = 2T_{n-2} - 2T_{n-1}X\,. \tag{234}$$

Equating the right-hand sides of (233) and (234), yields,

$$T_n = (X^{-1} + X)T_{n-1} - T_{n-2}\,, \quad \text{for } n \geq 2\,. \tag{235}$$

But, $X^{-1} + X = 2X_0$, yielding,

$$T_n(X_0) = 2X_0T_{n-1}(X_0) - T_{n-2}(X_0)\,, \quad \text{for } n \geq 2\,, \tag{236}$$

which is the recurrence relation for the Chebyshev polynomials of the first kind.

Now to the Cotes's connection to all this. On page 78, in his chapter on the roots of unity, Tignol recounts that Cotes found the formula

$$f_n(X) = 2a \qquad (n \text{ odd}) \,, \tag{237}$$

which Tignol explains that "$f_n(X)$ is the polynomial by which $2\cos n\alpha$ is expressed as a function of $2\cos\alpha$." As a result, the following recurrence relation for $f_n(X)$ was obtained:

$$f_{n+1}(X) = X f_n(X) - f_{n-1}(X) \,, \tag{238}$$

the proof of which was provided later by de Moivre. This recurrence relation is very close to that of the Chebyshev polynomials, lacking only a factor of 2 before the first term on the RHS. Perhaps we should refer to the polynomials generated by this recurrence as the 'Cotes Polynomials'.

But what are these polynomials? Tignol provides us with two equations by which we can answer this question:

$$1 + z^{2n} = f_n(X) z^n \,, \tag{239a}$$

where

$$X = z + z^{-1} \tag{239b}$$

(see pp. 78–79). Putting $n = 0$ into (239a), we get $f_0(X) = 2$. If we put $n = 1$ into it, we get $f_1(X) = z + z^{-1} = X$. And if we put $n = 2$ into it, we get $f_2(X) = z^2 + z^{-2} = X^2 - 2$. So far, the Cotes polynomials look a bit like the Lucas polynomials.

Putting (239a) into the explicit form

$$\begin{aligned} f_n(X) = z^n + z^{-n} &= (z + z^{-1})^n - \text{lower-order terms in } z \text{ and } z^{-1} \\ &= X^n - \text{lower-order terms in } z \text{ and } z^{-1} \,, \end{aligned} \tag{240}$$

it's easy to see $f_n(X)$ will be a monic polynomial in X for all $n \geq 0$.

Apparently, the Cotes polynomials share similar identities with the Chebyshev polynomials of the first kind (see p. 95), though with a difference of a factor of 2 predictably cropping up in places:

$$f_m(f_n(X)) = f_{mn}(X) \,, \tag{241a}$$

$$f_m(X) f_n(X) = f_{m+n}(X) + f_{|m-n|}(X) \,. \tag{241b}$$

I shouldn't be at all surprised if the Cotes polynomials are already known by some other name, but I haven't been able to find it, as yet.

Perhaps someone should take a closer look at the Cotes polynomials with an eye toward their practical applications. They may have a similar utility that the monic Chebyshev polynomials of the first kind have in the Min-Max Theorem.

Appendix 8

Show that if

$$P_n(X_0, X) \stackrel{\text{def}}{=} X^{2n} - 2T_n(X_0)X^n + 1\,, \tag{242}$$

then

$$P_n(\cosh\theta, X) = \prod_{k=0}^{n-1} P_1\left(\cosh\left(\theta + \frac{2k\pi i}{n}\right), X\right)\,. \tag{243}$$

Solution: First, note that

$$\begin{aligned} P_1\left(\cosh\left(\theta + \frac{2k\pi i}{n}\right), X\right) &= X^2 - 2X_0^{(k)}X + 1 \\ &= X^2 - 2\cosh\left(\theta + \frac{2k\pi i}{n}\right)X + 1\,. \end{aligned} \tag{244}$$

Ok,

$$\begin{aligned} P_n(X_0, X) &= X^{2n} - 2T_n(X_0)X^n + 1 \\ &= X^{2n} - 2b_0X^n + 1 \\ &= (X^n - b)(X^n - b^-) \\ &= \prod_{k=0}^{n-1}\big[X - \big(\sqrt[n]{|\,b_+\,|}\,\alpha^k u_+ + \sqrt[n]{|\,b_-\,|}\,\alpha^{-k}u_-\big)\big]\times \\ &\qquad\qquad \times\big[X - \big(\sqrt[n]{|\,b_+\,|}\,\alpha^k u_- + \sqrt[n]{|\,b_-\,|}\,\alpha^{-k}u_+\big)\big] \\ &= \prod_{k=0}^{n-1}\big[X^2 - \big(\sqrt[n]{|\,b_+\,|}\,\alpha^k + \sqrt[n]{|\,b_-\,|}\,\alpha^{-k}\big)X + \sqrt[n]{|\,b_+b_-\,|}\,\big] \\ &= \prod_{k=0}^{n-1}\big[X^2 - 2X_0^{(k)}X + 1\big] \\ &= \prod_{k=0}^{n-1} P_1\left(\cosh\left(\theta + \frac{2k\pi}{n}\right), X\right)\,, \end{aligned} \tag{245}$$

where we used that

$$\begin{aligned} X^{(k)} &= e^{u(i\theta + 2k\pi i/n)} \\ &= \sqrt[n]{b}\,\beta^k \\ &= \big(\sqrt[n]{|\,b_+\,|}\,u_+ + \sqrt[n]{|\,b_-\,|}\,u_-\big)\beta^k \\ &= \sqrt[n]{|\,b_+\,|}\,\alpha^k u_+ + \sqrt[n]{|\,b_-\,|}\,\alpha^{-k}\,u_-\,, \end{aligned} \tag{246}$$

to get

$$X_0^{(k)} \equiv \langle X^{(k)}\rangle\,, \tag{247}$$

or

$$\cosh(\theta + 2k\pi i/n) = \tfrac{1}{2}\big(\sqrt[n]{|\,b_+\,|}\,\alpha^k + \sqrt[n]{|\,b_-\,|}\,\alpha^{-k}\big)\,. \tag{248}$$

We can get a corollary of this in the complex algebra by setting $\theta = i\theta'$, $\theta' \in \mathbb{R}$, and $X = <X>$:

$$\boxed{P_n(\cos\theta', X) = \prod_{k=0}^{n-1} P_1\left(\cos\left(\theta' + \frac{2\pi k}{n}\right), X\right)} \tag{249}$$

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