The Hamiltonian-Jacobi Equation Using Structured Differentiation

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> We have done considerable mountain climbing [having just finished with the canonical transformation]. Now we are in the rarified atmosphere of theories of excessive beauty and we are nearing a high plateau on which geometry, optics, mechanics, and wave mechanics meet on common ground. Only concentrated thinking, and a considerable amount of re-creation, will reveal the full beauty of our subject in which the last word has not yet been spoken.
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> — Cornelius Lanczos[1]

Abstract

This paper is a follow-up to my previous paper "Canonical Generating Functions for Hamiltonians Using Structured Differentiation." This paper will also use SD, though its use is primarily already used up in the last paper. In reality, the Hamilton-Jacobi formulation of mechanics is itself just a special case of a canonical transformation in which we allow the Lagrangian to be explicitly dependent on time, in distinction to how it was treated in the last paper. Additionally, the generating function F_2 takes a prominent role, and it too is explictly a function of time. Hopefully, in time I'll improve the content of this paper.

1 Introduction

When I first studied Lagrangians, I was told that two Lagrangians, say, $\mathcal{L}$ and $\mathcal{L}'$ are the same[2] if they differ only by a total derivative of some scalar function, say F, to give

$$\mathcal{L} = \mathcal{L}' + \frac{dF}{dt}\,. \tag{1}$$

At that time, I did not realize the hidden importance of this claim. That would come much later, as I made the great effort to understand canonical transformations in Hamiltonian theory. It turns out that this total derivative is

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[1] Found in [1] on page 229, in the introduction to the chapter "The Partial Differential Equation of Hamilton-Jacobi."

[2] By the same, I mean they generate the same Euler-Lagrange system of equations.

a praega tallea[3] for in it hangs both canonical transformations with generating functions and the theory of Hamilton-Jacobi.

Type	Generating Function	Adjoint	Partials of F
1	$F_1 = F_1(q, Q, t)$	$-\dfrac{\partial P}{\partial q} = \dfrac{\partial p}{\partial Q}$	$\dfrac{\partial F_1}{\partial Q} = -P\,, \quad \dfrac{\partial F_1}{\partial q} = p$
2	$F_2 = F_2(q, P, t)$	$\dfrac{\partial Q}{\partial q} = \dfrac{\partial p}{\partial P}$	$\dfrac{\partial F_2}{\partial P} = Q\,, \quad \dfrac{\partial F_2}{\partial q} = p$
3	$F_3 = F_3(p, Q, t)$	$\dfrac{\partial P}{\partial p} = \dfrac{\partial q}{\partial Q}$	$\dfrac{\partial F_3}{\partial Q} = -P\,, \quad \dfrac{\partial F_3}{\partial p} = -q$
4	$F_4 = F_4(p, P, t)$	$\dfrac{\partial Q}{\partial p} = -\dfrac{\partial q}{\partial P}$	$\dfrac{\partial F_4}{\partial P} = Q\,, \quad \dfrac{\partial F_4}{\partial p} = -q$

Table 1. Generating Functions, Adjoints, and Partials of F. One use of this information is to decide if a proposed transformation is canonical or not.

And, by the way, if you completely understand the previous paper I wrote on canonical transformations with generating functions, this paper should be easy to follow — well, mostly.

Anyway, in the last paper, we considered the Lagrangians and Hamiltonians to be only implicitly dependent on time, not explicitly dependent on time. Therefore, last time, (1) reduced to

$$\mathcal{L} = \mathcal{L}' + \frac{\not\partial F}{\not\partial t}\,, \tag{2}$$

Let's take a moment to review how the total derivative works: The total derivative of F by time can be split into an explicit part $\partial F/\partial t$ and an implicit part $\not\partial F/\not\partial t$, by

$$\frac{dF}{dt} = \frac{\not\partial F}{\not\partial t} + \frac{\partial F}{\partial t}\,. \tag{3}$$

If we take $F = F_2$ according to the standard Type-2 transformation of coordinates of a canonical transformation as found in Table 1, then

$$F_2 = F_2(q, P, t)\,. \tag{4}$$

On differentiating this by time through q and P, but not through t explicitly, we get

$$\frac{\not\partial F_2}{\not\partial t} = \frac{\partial F_2}{\partial q}\dot{q} + \frac{\partial F_2}{\partial P}\dot{P}\,. \tag{5}$$

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[3] Laevendi for 'hidden gem'.

But from the second line of Table 1, we have that

$$\frac{\partial F_2}{\partial q} = p \qquad \text{and} \qquad \frac{\partial F_2}{\partial P} = Q\,, \tag{6}$$

therefore (5) becomes

$$\frac{\not\partial F_2}{\not\partial t} = p\dot{q} + Q\dot{P}\,. \tag{7}$$

But we also know that the Hamiltonian is a derived scalar function of a Lagrangian, given by

$$\mathcal{H} = \dot{q}p - \mathcal{L}\,, \tag{8a}$$

$$\mathcal{K} = \dot{Q}P - \mathcal{L}'\,, \tag{8b}$$

where $\mathcal{H}$ is the old Hamiltonian and $\mathcal{K}$ is the new Hamiltonian, under a transformation of coordinates. If we then turn things around, we have

$$\mathcal{L} = \dot{q}p - \mathcal{H}\,, \tag{9a}$$

$$\mathcal{L}' = \dot{Q}P - \mathcal{K}\,. \tag{9b}$$

So, if we substitute (9a), (9b), and (7) into (2), and then apply all such cancellations as we can find, we get

$$\mathcal{H} = \mathcal{K}\,. \tag{10}$$

Of course, this big simplification was not an accident, but was all by careful construction. In the previous paper, we continued by setting

$$E = \mathcal{H} = \mathcal{K}\,, \tag{11}$$

where E is the total energy of the system and it is a constant of the motion. From the algebraic viewpoint, this constant of the motion is not merely formal, but is a crucial algebraic constraint, used to find solutions to the problems induced by canonical transformations.

In the last paper, we went on to use those methods to solve the 1-D harmonic oscillator problem, using generating function $F_1 = F_1(q, Q)$:

$$\mathcal{H}(q,p) = \frac{p^2}{2m} + \frac{kq^2}{2} = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right)\,, \tag{12}$$

where $\omega^2 = \dfrac{k}{m}$. Although, as I said, we put $\partial F_1/\partial t = 0$.

2 Conversion to Proper Hamilton-Jacobi Form

This time, the Lagrangians and the Hamiltonians are permitted to be explicit functions of time, and that means that the explicit derivative we ignored in (1) to produce (2) must be put back in, and, keeping in mind that this time we are using the generating function F_2, the result is the simple change in (10) to

$$\mathcal{K} = \mathcal{H} + \frac{\partial F_2}{\partial t}\,, \tag{13}$$

where

$$F_2 = F_2(q, P, t)\,, \tag{14}$$

Now, apparently, F_2 is playing the role of the action S, so this last equation becomes

$$\mathcal{K} = \mathcal{H} + \frac{\partial S}{\partial t}\,, \tag{15}$$

where

$$S = S(q, P, t) \tag{16}$$

and is called *Hamilton's Principal Function.* We have the further refinements from (6), giving us

$$\frac{\partial S}{\partial q} = p \qquad \text{and} \qquad \frac{\partial S}{\partial P} = Q\,. \tag{17}$$

And let's not forget that we have performed a canonical transformation of variables, hence Hamilton's Equations are preserved in the new variables Q, P, giving us

$$\dot{Q} = \frac{\partial \mathcal{K}}{\partial P}\,, \tag{18a}$$

$$\dot{P} = -\frac{\partial \mathcal{K}}{\partial Q}\,. \tag{18b}$$

Now, we're going to do something a bit strange. Apparently we have the freedom to set

$$\mathcal{K} = 0\,, \tag{19}$$

which we seem to need, or at least some constraint on $\mathcal{K}$, because we no longer have the algebraic constraint in (11). Thus it seems that our system of equations is underdetermined prior to setting (19), which has the immediate consequence from (15) that

$$0 = \mathcal{H} + \frac{\partial S}{\partial t}\,, \tag{20}$$

which in some sense replaces (11).

From (18b), we get that P is a constant, which we'll refer to as α:

$$P = \alpha\,, \tag{21}$$

and therefore now

$$S = S(q, \alpha, t)\,. \tag{22}$$

Setting $\mathcal{K}$ to zero has forced a tighter functional dependency between $\mathcal{H}$ and $\frac{\partial S}{\partial t}$. And, if $\mathcal{H}$ happens not to be explicitly dependent on time, then we have no choice but to assume that $\dfrac{\partial S}{\partial t}$ has a linear function of time, and another term that's a function of q and α but not t. Accordingly, we let

$$S = S(q,\alpha,t) = W(q,\alpha) - Et\,, \tag{23}$$

where $W(q,\alpha)$ is referred to as *Hamilton's Characteristic Function*; and E, the total energy, entered the picture here because $\mathcal{H} = E$. And for our next bit of legerdomian, we set $E = \alpha$, which is done in the literature I can find on the subject, but it's not clear yet to me why we can do this. Hence,

$$S = S(q,\alpha,t) = W(q,\alpha) - \alpha t\,. \tag{24}$$

And from (18b), we get that Q is a constant, which we'll refer to as β:

$$Q = \beta\,. \tag{25}$$

But wait! There's more!

$$Q = \beta = \frac{\partial S}{\partial P} = \frac{\partial S}{\partial \alpha}\,. \tag{26}$$

So, instead of solving for

$$q = q(P,Q,t) \qquad \text{and} \qquad p = p(P,Q,t)\,, \tag{27}$$

we're now solving for

$$q = q(\alpha,\beta,t) \qquad \text{and} \qquad p = p(\alpha,\beta,t)\,, \tag{28}$$

Well, not quite! What we really want to do is to set $\alpha = p_0$ and $\beta = q_0$, where q_0 and p_0 are, respectively, the initial position and momentum of the system. Now,

$$q = q(p_0,q_0,t) \qquad \text{and} \qquad p = p(p_0,q_0,t)\,. \tag{29}$$

A Royal Fizzbin! Anyway, let's see if we can put this stuff to work.

3 Harmonic Oscillator in 1-D Solved

The Hamiltonian for the 1-dimensional harmonic oscillator is given by

$$\mathcal{H}(q,p) = \frac{p^2}{2m} + \frac{kq^2}{2} = \frac{1}{2m}\big(p^2 + m^2\omega^2 q^2\big)\,, \tag{30}$$

where $\omega^2 = \dfrac{k}{m}$.

Using that $p = \dfrac{\partial S}{\partial q}$ from (17) put into (20), we get

$$\begin{aligned} 0 &= \frac{1}{2m}\big(p^2 + m^2\omega^2 q^2\big) + \frac{\partial S}{\partial t} \\ &= \frac{1}{2m}\bigg[\bigg(\frac{\partial S}{\partial q}\bigg)^2 + m^2\omega^2 q^2\bigg] + \frac{\partial S}{\partial t}\,. \end{aligned} \tag{31}$$

Now, using (24), we have that

$$\frac{\partial S}{\partial t} = -\alpha \qquad \text{and} \qquad \frac{\partial S}{\partial q} = \frac{\partial W}{\partial q}\,, \tag{32}$$

therefore,

$$\frac{1}{2m}\Big[\Big(\frac{\partial W}{\partial q}\Big)^2 + m^2\omega^2 q^2\Big] = \alpha\,. \tag{33}$$

On solving this for W we get

$$W = \sqrt{2m\alpha}\int\Big[1 - \frac{m\omega^2 q^2}{2\alpha}\Big]^{1/2} dq\,. \tag{34}$$

And since from (24), we have that $S = W - \alpha t$, then

$$S = \sqrt{2m\alpha}\int\Big[1 - \frac{m\omega^2 q^2}{2\alpha}\Big]^{1/2} dq - \alpha t\,. \tag{35}$$

Now, from (17) with its variable changes, and with a bit of algebra, we get

$$\beta = \frac{\partial S}{\partial \alpha} = \sqrt{\frac{m}{2\alpha}}\int\frac{dq}{\sqrt{1 - \dfrac{m\omega^2 q^2}{2\alpha}}} - t\,. \tag{36}$$

Therefore

$$t + \beta = \frac{1}{\omega}\sin^{-1}\Big(q\sqrt{\frac{m\omega^2}{2\alpha}}\Big)\,. \tag{37}$$

Solving for q, we get

$$q = \sqrt{\frac{2\alpha}{m\omega^2}}\sin\big(\omega t + \beta_\omega\big)\,, \tag{38}$$

where $\beta_\omega = \beta\omega$. And, of course,

$$p = \frac{\partial S}{\partial q} = \frac{\partial W}{\partial q} = \sqrt{2m\alpha - m^2\omega^2 q^2}\,. \tag{39}$$

On using (38), we get a trigonometric form like that of (38) for q:

$$p = \sqrt{2m\alpha}\cos\big(\omega t + \beta_\omega\big)\,. \tag{40}$$

References

[1] C. Lanczos, *The Variational Principles of Mechanics*, 4th ed. Dover Publications (1970).