Structured Differentiation for Advanced Calculus

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> Matters of notation play a
> considerable role in connection with
> the chain rule. Wide varieties
> of usage exist in mathematical writing
> where the chain rule is concerned.
>
> —Taylor & Mann

Introduction

The purpose of this paper is to present a structured, semantically unified formalism for differentiation to meet the needs of the undergraduate and graduate mathematics student. The problems in this paper are taken mostly from texts on advanced calculus. Formally, we shall refer to the formalism used herein as *structured differentiation* or SD. In particular, this article is a continuation of the article "A structured differentiation for physicists," published in the *AJNP*[1] in January of 1992, which should be take as a reference to the notation used in this article. The current article is an expansion of the article I published in the *AJNP* in April 1996.

Review of some important issues

Let's start by presenting a notational ambiguity presented by Buck ([1] p. 137–8). He refers to the ambiguity of partial derivatives. Let

$$w = f(x,u,v) \qquad u = g(x,v,y) \qquad v = h(x,y)\,. \tag{1}$$

For the derivative of w by x Buck offers:

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial v}\frac{\partial v}{\partial x} \tag{2}$$

Without further notation this equation is ambiguous. There are a number of solutions. One is to subscript $\frac{\partial w}{\partial x}$ on the right to get $\left(\frac{\partial w}{\partial x}\right)_y$, meaning that

[1] The *Arizona Journal of Natural Philosophy*.

y is held constant during the differentiation. If we do this then we interpret $\dfrac{\partial w}{\partial x}$ on the left as the "total" partial derivative, and $\left(\dfrac{\partial w}{\partial x}\right)_y$ on the right as the "partial" partial derivative. More to the liking of authors Taylor & Mann ([2]) is to re-express (2) as

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial h}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial g}{\partial v}\frac{\partial h}{\partial x} \tag{3}$$

But this is still very unsatisfactory, so I think that Taylor & Mann would probably also introduce the primitive function $W(x, y) = f(x, v, y)$, and thus write

$$\frac{\partial W}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial h}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial g}{\partial v}\frac{\partial h}{\partial x} \tag{4}$$

At least at this point we know exactly how to interpret $\dfrac{\partial W}{\partial x}$ because $W(x, y)$ is primitive in x. $\dfrac{\partial W}{\partial x}$ must be an explicit derivative and a total derivative. But this still leaves ambiguity in $\dfrac{\partial g}{\partial x}$ ($\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial h}{\partial x}$ must be total/explicit derivatives). So, I think that we should also introduce the primitive function $G(x, y) = g(x, v, y)$, and so the equation becomes

$$\frac{\partial W}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial G}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial h}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial g}{\partial v}\frac{\partial h}{\partial x} \tag{5}$$

At least by doing all these contortions we have removed the ambiguities. Now all the partial derivatives can be interpreted as explicit derivatives. And it should be remembered that the canonical form for displaying answers involving total derivatives is to represent them as explicit derivatives regardless of the formalism chosen.

But there is an easier way to do this derivative. Instead of overloading the partial derivative to be all things for all occasions, why not just define two other derivative symbols? Thus in SD we have the partial derivative meaning explicit derivative, the copartial derivative $\not\partial$ meaning implicit derivative, and the deltal derivative δ meaning the total derivative, and the three operators satisfy the equation

$$\frac{\delta}{\delta x} = \frac{\partial}{\partial x} + \frac{\not\partial}{\not\partial x} \tag{6}$$

which can also be written as

$$\delta_x = \partial_x + \not\partial_x \tag{7}$$

and which can be further simplified to

$$\delta = \partial + \not\partial\,. \tag{8}$$

With this complete differential operator we can operate on $w = w(x, u, v)$ to get

$$\frac{\delta w}{\delta x} = \frac{\partial w}{\partial x} + \frac{\not\partial w}{\not\partial x} \tag{9}$$

or,

$$
\begin{aligned}
\frac{\delta w}{\delta x} &= \frac{\partial w}{\partial x} + \frac{\partial w}{\partial u}\frac{\delta u}{\delta x} + \frac{\partial w}{\partial v}\frac{\delta v}{\delta x} \\
&= \frac{\partial w}{\partial x} + \frac{\partial w}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial v}\frac{\delta v}{\delta x}\right) + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} \\
&= \frac{\partial w}{\partial x} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}\,.
\end{aligned}
\tag{10}
$$

Now, for the derivative of w by y, we get, given that $\partial w/\partial y \equiv 0$

$$
\frac{\delta w}{\delta y} = \frac{\not\partial w}{\not\partial y} = \frac{\partial w}{\partial u}\frac{\delta u}{\delta y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial w}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial v}\frac{\partial v}{\partial y}\right) + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}\,. \tag{11}
$$

As a final comment, the deltal derivative reduces to an ordinary derivative when the function being differentiated is a function of only one independent (fundamental) variable.

But Taylor & Mann have their problems trying to present a consistent notation and vocabulary too. On page 271 we find: Consider the function $G(x, y)$ as a function of u and y, with $x = f(u, y)$. The partial derivative with respect to y is

$$
\frac{\partial G}{\partial x}\frac{\partial f}{\partial y} + \frac{\partial G}{\partial y} = \frac{\left|\dfrac{\partial(F, G)}{\partial(x, y)}\right|}{\dfrac{\partial F}{\partial x}} = 0\,, \tag{12}
$$

where I have inserted the determinant symbols to conform to SD. As seen from the perspective of SD the phrase "partial derivative with respect to y" is bizarre. First, because it doesn't even bother to mention what is being differentiated! Of course it's G that's being differentiated, but the left-hand side of this equation is not a partial derivative, it's a total derivative. In any case, the symbol $\partial G/\partial y$ is just a term on the left-hand side.

In SD the phrase "partial derivative" has only one symbol, namely ∂, and it has only one meaning, namely, it is an explicit derivative. It is probably safe to say that the only way to make sense of the standard notations/symbology of "partial differentiation" is to already know what it means. Even the term "partial differentiation" is a misnomer—it's really "total differentiation."

On page 175, problem 22 we find the problem "If $u = x^2 + y^2 + z^2$ and $z = xyv$, how many meanings are there for $\partial u/\partial y$?" Well, fortunately in SD it has only one!

Relation to the chain rule:

Before doing the solved problems, I want to show that the parametric split of the delta derivative into

$$
\frac{\delta}{\delta x} = \frac{\partial}{\partial x} + \frac{\not\partial}{\not\partial x}
$$

is consistent with the ordinary chain rule. Let $f = f(x, y, z(x, y))$ and let $\mathbf{x} = (x, y, z)^t$. Then, by the chain rule

$$\frac{\delta f}{\delta x} = \frac{\partial f}{\partial \mathbf{x}} \frac{\delta \mathbf{x}}{\delta x} = \frac{\partial f}{\partial x} \frac{\delta x}{\delta x} + \frac{\partial f}{\partial (y, z)} \frac{\delta (y, z)}{\delta x} = \frac{\partial f}{\partial x} + \frac{\not\partial f}{\not\partial x} \,. \tag{13}$$

Solved problems from Taylor & Mann

PROBLEM:

On page 190 problem 3 we find (in non-SD notation): If $z = f(x, y)$ is a solution of $F(x, y, z) = 0$ (with $F_{,3} \neq 0$), and if $H(x, y) = G(x, y, f(x, y))$, show that

$$\frac{\partial F}{\partial z} \frac{\partial H}{\partial y} = -\frac{\partial (F, G)}{\partial (y, z)} \,, \tag{14}$$

where $F_{,3}$ means that F is explicitly differentiated by z, the third variant in the variant list of F.

SOLUTION:

(This is one annoyance in SD: We have to specifically declare that the quantity in the right-hand-side of (14) is a determinant, though in standard advanced calculus this is unnecessary. The reason it needs to be included in SD is because the notation $\partial(\cdots)/\partial(\cdots)$ is in general an $m \times n$ matrix. I believe that the advantages of this notation far outweigh its disadvantages.)

First, we really don't need to introduce H, which is just the primitive rewrite of G. If we ignore H for now we can rewrite (14) in SD notation as

$$\frac{\partial F}{\partial z} \frac{\delta G}{\delta y} = -\left| \frac{\partial (F, G)}{\partial (y, z)} \right| \,. \tag{15}$$

Our starting point is the equation

$$F(x, y, z(x, y)) = F(\boldsymbol{\eta}, z(\boldsymbol{\eta})) = 0 \,. \tag{16}$$

where $\boldsymbol{\eta} \equiv (x, y)^t$ and the raised t means transpose. We are purposely keeping the notation consistent with a matrix formulation since we will be going in and out of matrix representations as the situation calls. On differentiating (16) by $\boldsymbol{\eta}$ we get

$$\frac{\delta F(\boldsymbol{\eta}, z(\boldsymbol{\eta}))}{\delta \boldsymbol{\eta}} = \frac{\partial F}{\partial \boldsymbol{\eta}} + \frac{\partial F}{\partial z} \frac{\delta z}{\delta \boldsymbol{\eta}} = \frac{\partial F}{\partial \boldsymbol{\eta}} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial \boldsymbol{\eta}} = 0 \tag{17}$$

from which we get

$$\frac{\partial z}{\partial \boldsymbol{\eta}} = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right) = -{F_{,z}}^{-1} \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right) \,. \tag{18}$$

Now

$$\frac{\delta G}{\delta \boldsymbol{\eta}} = \frac{\partial G}{\partial \boldsymbol{\eta}} + \frac{\partial G}{\partial z}\frac{\partial z}{\partial \boldsymbol{\eta}} = \frac{\partial G}{\partial \boldsymbol{\eta}} - F_{,z}{}^{-1}\,\frac{\partial G}{\partial z}\frac{\partial F}{\partial \boldsymbol{\eta}}\,. \tag{19}$$

Multiplying through by $F_{,z} = \partial F/\partial z$ we get

$$\frac{\partial F}{\partial z}\frac{\delta G}{\delta \boldsymbol{\eta}} = \left(\frac{\partial G}{\partial x}\frac{\partial F}{\partial z} - \frac{\partial F}{\partial x}\frac{\partial G}{\partial z}, \frac{\partial G}{\partial y}\frac{\partial F}{\partial z} - \frac{\partial F}{\partial y}\frac{\partial G}{\partial z}\right) = \left(-\frac{\partial(F,G)}{\partial(x,z)}, -\frac{\partial(F,G)}{\partial(y,z)}\right)\,. \tag{20}$$

And the desired result is the second component of the above.

Notice that when going from the compact notation of vector symbols to matrix formulation, we must be very careful to put the matrices in the correct order so that the multiplication is both well-defined and consistent.

LEMMA 1

Since the deltal derivative is a generalized total derivative, we can take the total derivative of a vector-valued function of a vector variable. Let's do that now for a specific form of functional dependence. Let the function $\mathbf{f}(\mathbf{u}, \mathbf{v}) = 0$, where $\mathbf{u}$ is the fundamental of $\mathbf{f}$ and $\mathbf{v} = \mathbf{v}(\mathbf{u})$; then

$$\frac{\delta \mathbf{f}}{\delta \mathbf{u}} = \frac{\partial \mathbf{f}}{\partial \mathbf{u}} + \frac{\partial \mathbf{f}}{\partial \mathbf{v}}\frac{\delta \mathbf{v}}{\delta \mathbf{u}} = 0\,. \tag{21}$$

Therefore, if $\partial \mathbf{f}/\partial \mathbf{v}$ is assumed to be an invertible square matrix, we get

$$\frac{\delta \mathbf{v}}{\delta \mathbf{u}} = -\left(\frac{\partial \mathbf{f}}{\partial \mathbf{v}}\right)^{-1}\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\,. \tag{22}$$

If $\mathbf{v}$ happens to be primitive with respect to $\mathbf{u}$ then this last equation can be written as

$$\frac{\partial \mathbf{v}}{\partial \mathbf{u}} = -\left(\frac{\partial \mathbf{f}}{\partial \mathbf{v}}\right)^{-1}\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\,. \tag{23}$$

PROBLEM:

From page 191 , problem 4 reads: $u = f(x, y, z)$, $v = g(x, y, z)$ are solutions of $F(x, y, z, u, v) = 0$, $G(x, y, z, u, v) = 0$. Let $K(x, y, z) = H(x, y, z, f(x, y, z), g(x, y, z))$. Show that (under suitable conditions)

$$\frac{\partial K}{\partial z} = \frac{\left|\dfrac{\partial(F,G,H)}{\partial(z,u,v)}\right|}{\left|\dfrac{\partial(F,G)}{\partial(u,v)}\right|}\,. \tag{24}$$

SOLUTION:

We know how to interpret $\partial K/\partial z$ because K is the primitive reduction of H — that makes the partial here an explicit derivative just as in SD, but K was only introduced so that the explicit derivative would also be a total derivative. But SD has a total derivative operator, so we will dispense with K and instead solve for $\delta H/\delta z$, since it is equal to $\partial K/\partial z$.

First, we let

$$\mathbf{F} \equiv (F, G)^t = 0\,, \tag{25}$$

with $\mathbf{x} \equiv (x, y, z)^t$ and $\mathbf{u} \equiv \mathbf{u}(\mathbf{x})$, then the derivative becomes

$$\frac{\delta \mathbf{F}}{\delta \mathbf{x}} = \frac{\partial \mathbf{F}}{\partial \mathbf{x}} + \frac{\partial \mathbf{F}}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial \mathbf{x}} = 0\,, \tag{26}$$

so from Lemma 1 we get

$$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} = -\left(\frac{\partial \mathbf{F}}{\partial \mathbf{u}}\right)^{-1} \frac{\partial \mathbf{F}}{\partial \mathbf{x}}\,. \tag{27}$$

This result will be substituted in this next equation:

$$\frac{\delta H}{\delta \mathbf{x}} = \frac{\partial H}{\partial \mathbf{x}} + \frac{\partial H}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial \mathbf{x}}\,, \tag{28}$$

to get

$$\begin{aligned} \frac{\delta H}{\delta \mathbf{x}} &= \frac{\partial H}{\partial \mathbf{x}} + \frac{\partial H}{\partial \mathbf{u}} \left[-\left(\frac{\partial \mathbf{F}}{\partial \mathbf{u}}\right)^{-1} \frac{\partial \mathbf{F}}{\partial \mathbf{x}} \right] \\ &= \left(\frac{\partial H}{\partial \mathbf{x}} \left| \frac{\partial \mathbf{F}}{\partial \mathbf{u}} \right| - \frac{\partial H}{\partial \mathbf{u}} \operatorname{adj}\left[\frac{\partial \mathbf{F}}{\partial \mathbf{u}}\right] \frac{\partial \mathbf{F}}{\partial \mathbf{x}} \right) \left| \frac{\partial \mathbf{F}}{\partial \mathbf{u}} \right|^{-1}\,. \end{aligned} \tag{29}$$

Upon expanding this and then focusing on only the z component, we get

$$\begin{aligned} \frac{\partial K}{\partial z} = \frac{\delta H}{\delta z} &= \left(\left| \frac{\partial(F,G)}{\partial(u,v)} \right| \frac{\partial H}{\partial z} - \left| \frac{\partial(H,G)}{\partial(u,v)} \right| \frac{\partial F}{\partial z} + \left| \frac{\partial(H,F)}{\partial(u,v)} \right| \frac{\partial G}{\partial z} \right) \left| \frac{\partial \mathbf{F}}{\partial \mathbf{u}} \right|^{-1} \\ &= \frac{\left| \dfrac{\partial(F,G,H)}{\partial(z,u,v)} \right|}{\left| \dfrac{\partial(F,G)}{\partial(u,v)} \right|}\,. \end{aligned} \tag{30}$$

PROBLEM:

Again from page 191, problem 6: If the equation $F(x, y, z) = 0$ can be solved for each one of the variables in terms of the other two, show, by taking for granted certain general conditions, that

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1\,. \tag{31}$$

SOLUTION:

Treating $y = y(x, z)$ and taking the total derivative of F by x yields

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} = 0\,. \tag{32}$$

Treating $z = z(x, y)$ and taking the total derivative of F by y yields

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0\,. \tag{33}$$

Treating $x = x(y, z)$ and taking the total derivative of F by z yields

$$\frac{\partial F}{\partial z} + \frac{\partial F}{\partial x}\frac{\partial x}{\partial z} = 0\,. \tag{34}$$

Now these last three equations can be put in matrix form, yielding

$$\begin{pmatrix} 1 & \frac{\partial y}{\partial x} & 0 \\ 0 & 1 & \frac{\partial z}{\partial y} \\ \frac{\partial x}{\partial z} & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial y} \\ \frac{\partial F}{\partial z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\,. \tag{35}$$

Since this is a homogeneous matrix equation, the determinant of the 3×3 matrix is zero, yielding

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = -1 \tag{36}$$

where the subscripts are unnecessary if you keep tract of the dependencies.

PROBLEM:

And on page 192, problem 13: If

$$z = \phi(x, y) \tag{37}$$

satisfies the equation $F(f(x, y, z), g(x, y, z)) = 0$, show that

$$\frac{\partial z}{\partial y} = -\frac{F_1 f_2 + F_2 g_2}{F_1 f_3 + F_2 g_3}\,. \tag{38}$$

Remember that in the notation of Taylor & Mann, F_1 means the partial derivative of F by its first variant, namely, f, which we'll call u, and f_2 means the partial derivative of f by its second variant y, etc. In our current notation, the above should be written as

$$\frac{\partial z}{\partial y} = -\frac{F_{,1}\, f_{,2} + F_{,2}\, g_{,2}}{F_{,1}\, f_{,3} + F_{,2}\, g_{,3}}\,. \tag{39}$$

NOTE: SD uses a subscripting notation similar to most versions of old-fashion tensor calculus. The widget h_1 refers to the first component of the "vector" h_i, or $\mathbf{h}$. The widget $h_{i,j}$ refers to the jth derivative by some variable of the ith component of the "vector" h_i. For example, $h_{i,j}$ could refer to the expanded form $\frac{\partial h_i}{\partial x_j}$.

SOLUTION:

First we need to rewrite all this stuff in a more compact form. So let's set $z = z(x, y)$, $\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, so our main equation becomes

$$F(\mathbf{u}(\mathbf{x}, z(\mathbf{x})) = 0\,. \tag{40}$$

On taking the total derivative of F by $\mathbf{x}$ we get

$$\frac{\partial F}{\partial \mathbf{u}} \frac{\delta \mathbf{u}}{\delta \mathbf{x}} = \frac{\partial F}{\partial \mathbf{u}} \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \frac{\partial \mathbf{u}}{\partial z} \frac{\partial z}{\partial \mathbf{x}} \right) = 0\,. \tag{41}$$

On expanding this we get

$$\left(\frac{\partial F}{\partial u}, \frac{\partial F}{\partial v} \right) \left[\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} + \begin{pmatrix} \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial z} \end{pmatrix} \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right) \right] = \big(0\,, 0\big)\,. \tag{42}$$

On distributing the product we get

$$\left(\frac{\partial F}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial F}{\partial v} \frac{\partial v}{\partial x}, \frac{\partial F}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial F}{\partial v} \frac{\partial v}{\partial y} \right) + \left(\frac{\partial F}{\partial u} \frac{\partial u}{\partial z} + \frac{\partial F}{\partial v} \frac{\partial v}{\partial z} \right) \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right) = \big(0\,, 0\big)\,. \tag{43}$$

From the first component of this we get

$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial u} \dfrac{\partial u}{\partial x} + \dfrac{\partial F}{\partial v} \dfrac{\partial v}{\partial x}}{\dfrac{\partial F}{\partial u} \dfrac{\partial u}{\partial z} + \dfrac{\partial F}{\partial v} \dfrac{\partial v}{\partial z}}\,. \tag{44}$$

The solution for the partial derivative $\frac{\partial z}{\partial y}$ is left to the reader.

Assorted problems from various sources

PROBLEM:

Let $w(\mathbf{x}) = w(r(\mathbf{x}), \phi(\mathbf{x}))$ be given by $\mathbf{x} = (x, y)^t$ and

$$\begin{cases} x = r \cosh \phi \\ y = r \sinh \phi \end{cases}\,. \tag{45}$$

Find the derivatives of w with respect to x, y in terms of derivative in terms of r, ϕ.

SOLUTION:

There are two simple ways to do this. In both cases we wish to write the equation

$$\frac{\partial w}{\partial \mathbf{x}} = \frac{\partial w}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial \mathbf{x}}, \tag{46}$$

where $\mathbf{u} = (r(\mathbf{x}), \phi(\mathbf{x}))^t$. So what we need to know is the transformation matrix $\frac{\partial \mathbf{u}}{\partial \mathbf{x}}$. The first way to get it to write the equation $\mathbf{x} = \mathbf{x}(\mathbf{u}(\mathbf{x}))$. Then the total derivative of this by $\mathbf{x}$ gives us

$$1 = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \mathbf{u}}, \tag{47}$$

from which we get

$$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \left(\frac{\partial \mathbf{x}}{\partial \mathbf{u}} \right)^{-1}. \tag{48}$$

The second way is to virtually emplace $\mathbf{F} = 0$ by the equations

$$\begin{cases} F_1 = x - r\cosh\phi \\ F_2 = y - r\sinh\phi \end{cases} \tag{49}$$

where the subscripts are not indicating partial derivatives. Then we just use the formula (23). In either case we end up with

$$\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \right) = \left(\frac{\partial w}{\partial r}, \frac{\partial w}{\partial \phi} \right) \begin{pmatrix} \cosh\phi & -\sinh\phi \\ \frac{-\sinh\phi}{r} & \frac{\cosh\phi}{r} \end{pmatrix}, \tag{50}$$

which, of course, is valid where r is different from zero.

DEFINITION: A *convolution* is said to occur whenever a variable or function is dependent on itself (this is not directly related to convolution in Laplace transform theory).

EXAMPLE:

For an example, consider the equation

$$f(f(x, y), y) = 0. \tag{51}$$

Now say that we want to solve for y as a function of x about some point p. We may derive yet another condition on f by differentiating (51) by y.

$$\frac{\delta f}{\delta y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial x} \frac{\partial x}{\partial y} = 0, \tag{52}$$

but since $x = f(x, y)$ then $\partial x/\partial y = \partial f/\partial y$, so

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} = 0\,. \tag{53}$$

Factoring, we get

$$\frac{\partial f}{\partial y}\left(1 + \frac{\partial f}{\partial x}\right) = 0\,. \tag{54}$$

But at p, $\partial f/\partial y \neq 0$ by assumption, therefore

$$\frac{\partial f}{\partial x} = -1\,. \tag{55}$$

PROBLEM:

Given a surface defined by $F(x, y, z) = 0$, we know that if $\partial F/\partial z \neq 0$ then, by the Implicit Function Theorem, $z = z(x, y)$. Show that the direction of the normal to the surface is given by the direction ratios

$$\frac{\partial z}{\partial x} \quad : \quad \frac{\partial z}{\partial y} \quad : \quad -1\,. \tag{56}$$

SOLUTION:

From vector calculus we know that at any point on the surface defined by the locus of points $F(x, y, z) = 0$ where the normal is defined, the direction of the normal to the surface is given by the gradient $\partial F/\partial \mathbf{x}$. Now, note that since $F = 0$, then

$$\frac{\delta F}{\delta x} = \frac{\partial F}{\partial x} + \frac{\not\partial F}{\not\partial x} = 0\,, \quad \frac{\delta F}{\delta y} = \frac{\partial F}{\partial y} + \frac{\not\partial F}{\not\partial y} = 0 \tag{57}$$

from which we conclude that

$$\frac{\partial F}{\partial x} = -\frac{\not\partial F}{\not\partial x}\,, \quad \frac{\partial F}{\partial y} = -\frac{\not\partial F}{\not\partial y} \tag{58}$$

Thus,

$$\begin{aligned}
\frac{\partial F}{\partial \mathbf{x}} &= \left(\frac{\partial F}{\partial x}\,,\ \frac{\partial F}{\partial y}\,,\ \frac{\partial F}{\partial z}\right) \\
&= \left(-\frac{\not\partial F}{\not\partial x}\,,\ -\frac{\not\partial F}{\not\partial y}\,,\ \frac{\partial F}{\partial z}\right) \\
&= -\left(\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}\,,\ \frac{\partial F}{\partial z}\frac{\partial z}{\partial y}\,,\ -\frac{\partial F}{\partial z}\right) \\
&= -\frac{\partial F}{\partial z}\left(\frac{\partial z}{\partial x}\,,\ \frac{\partial z}{\partial y}\,,\ -1\right)
\end{aligned} \tag{59}$$

Therefore the direction ratios of the normal to the surface is given by (56).

PROBLEM:

Given a surface defined by

$$\begin{cases} x = x \\ y = y \\ z = z(x, y) \end{cases} \tag{60}$$

the direction of the normal to the surface is given by the direction ratios

$$\frac{\partial z}{\partial x} \quad : \quad \frac{\partial z}{\partial y} \quad : \quad -1 \,. \tag{61}$$

Now, note that a surface in 3-space needs only two intrinsic variables to specify coordinates on it. Thus we can then rewrite the coordinates of the surface in the 3d embedding space as

$$\begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases} \tag{62}$$

Show that the direction ratios of the normal can be given by

$$j_1 \quad : \quad j_2 \quad : \quad j_3 \tag{63}$$

where

$$j_1 = \left| \frac{\partial(y, z)}{\partial(u, v)} \right| \quad : \quad j_2 = \left| \frac{\partial(z, x)}{\partial(u, v)} \right| \quad : \quad j_3 = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \,. \tag{64}$$

SOLUTION:

As always in this type of problem, the solution is most easily perceived only after make the standard simplifications. One of these is to write the "state" vector, $\mathbf{x} = (x, y, z)^t$. Another is to determine the before and after fundamentals: The before fundamental is $\boldsymbol{\eta} = (x, y)^t$ and the after is $\boldsymbol{\eta}' = (u, v)^t$. So, we can write

$$\mathbf{x}(\boldsymbol{\eta}) = \mathbf{x}(\boldsymbol{\eta}'(\boldsymbol{\eta})) \,. \tag{65}$$

Note that the two variables x, y are convoluted through $\boldsymbol{\eta}'$ to $\boldsymbol{\eta}$. The jacobian J_T of the transformation

$$\boldsymbol{\eta}' = T(\boldsymbol{\eta}) = \boldsymbol{\eta}'(\boldsymbol{\eta}) \tag{66}$$

is then

$$J_T = \left| \frac{\partial \boldsymbol{\eta}'}{\partial \boldsymbol{\eta}} \right| = j_3^{-1} \,. \tag{67}$$

Differentiating (65) and simplifying the total derivatives to partials we get

$$\frac{\delta \mathbf{x}}{\delta \boldsymbol{\eta}} = \frac{\partial \mathbf{x}}{\partial \boldsymbol{\eta}'} \frac{\partial \boldsymbol{\eta}'}{\partial \mathbf{x}} = \frac{\partial \mathbf{x}}{\partial \boldsymbol{\eta}'} \, \mathbf{j}_3^{-1} \tag{68}$$

On writing this last equation in matrix form and simplifying we get

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \dfrac{\partial z}{\partial x} & \dfrac{\partial z}{\partial y} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial u}{\partial v} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} \end{pmatrix} \left(\mathbf{j}_3^{-1}\right). \tag{69}$$

Now, we need to get determinant-form information from this, but how do we do this when we don't have square matrices? Actually, the square matrices are in (69) – there are three of them – and we can solve the problem from here by inspection. But let's prove that this is legal. Notice that the result of multiplying the 3×2 matrix by the 2×2 matrix on the right of (69) (to produce another 3×2 matrix) does not intermix the elements of the rows of the original 3×2 matrix. In other words, the result of the multiplication of the two matrices produces a matrix whose any two rows could be calculated without the presence of the third row — even in the original 3×2 matrix. So, we can decompose (69) into three equations of 2×2 matrices from which we can take determinants.

Let's do that now. Taking the first two rows we get

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial u}{\partial v} \end{pmatrix} \left(\mathbf{j}_3^{-1}\right). \tag{70}$$

From rows 1 and 3 we get

$$\begin{pmatrix} 1 & 0 \\ \dfrac{\partial z}{\partial x} & \dfrac{\partial z}{\partial y} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} \end{pmatrix} \left(\mathbf{j}_3^{-1}\right). \tag{71}$$

And, finally, from rows 2 and 3 we get

$$\begin{pmatrix} 0 & 1 \\ \dfrac{\partial z}{\partial x} & \dfrac{\partial z}{\partial y} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial y}{\partial u} & \dfrac{\partial u}{\partial v} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} \end{pmatrix} \left(\mathbf{j}_3^{-1}\right). \tag{72}$$

From the determinant of (70) we get

$$1 = j_3\, j_3^{-1}\,. \tag{73}$$

From the determinant of (71) we get

$$\frac{\partial z}{\partial y} = -j_2 j_3^{-1}\,. \tag{74}$$

From the determinant of (72) we get

$$\frac{\partial z}{\partial x} = -j_1 j_3^{-1} \,. \tag{75}$$

So, on substituting these values into (61) we get

$$-j_1/j_3 \quad : \quad -j_2/j_3 \quad : \quad -1 \tag{76}$$

which is equivalent to

$$j_1 \quad : \quad j_2 \quad : \quad j_3 \,. \tag{77}$$

Now I'll justify how I decompose a 3×2 matrix down to a 3×2 matrix by a particular example. To arrive at Eq. (71), left multiply Eq. (69) by the 2×3 matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{78}$$

to get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial u}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \left(\mathbf{j}_3^{-1}\right) , \tag{79}$$

and, using the fact that matrix algebra is associative, then, on simplification, the result will be Eq. (71).

PROBLEM:

Let $\mathbf{x} = \mathbf{x}(\mathbf{u})$ where $\mathbf{x}$ is the new fundamental, $\mathbf{u}$ is the old fundamental, $\mathbf{x} = (x, y, z)^t$, $\mathbf{u} = (u, v, w)^t$, show that

$$\frac{\partial x}{\partial u} = \left| \frac{\partial(v, w)}{\partial(y, z)} \right| \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| \,. \tag{80}$$

SOLUTION:

There are likely many good ways to solve this problem. I'll present just one. Let us form the state vector

$$\Psi = \begin{pmatrix} x \\ v \\ w \end{pmatrix} \,. \tag{81}$$

Now let's introduce some notation. Let $A((\mathbf{x}))$ mean that $\mathbf{x}$ is the fundamental of A, though it may not be the variant of A. Thus, we can write the two parametrizations of Ψ as

$$\Psi((\mathbf{u})) = \begin{pmatrix} x((\mathbf{u})) \\ v \\ w \end{pmatrix} , \qquad \Psi((\mathbf{x})) = \begin{pmatrix} x \\ v((\mathbf{x})) \\ w((\mathbf{x})) \end{pmatrix} \,. \tag{82}$$

Now we take the total derivative of $\Psi((\mathbf{u})) = \Psi(\mathbf{x}(\mathbf{u}))$ by $\mathbf{u}$ to get

$$\begin{pmatrix} \frac{\delta x}{\delta u} & \frac{\delta x}{\delta v} & \frac{\delta x}{\delta w} \\ \frac{\delta v}{\delta u} & \frac{\delta v}{\delta v} & \frac{\delta v}{\delta w} \\ \frac{\delta w}{\delta u} & \frac{\delta w}{\delta v} & \frac{\delta w}{\delta w} \end{pmatrix} = \begin{pmatrix} \frac{\delta x}{\delta x} & \frac{\delta x}{\delta y} & \frac{\delta x}{\delta z} \\ \frac{\delta v}{\delta x} & \frac{\delta v}{\delta y} & \frac{\delta v}{\delta z} \\ \frac{\delta w}{\delta x} & \frac{\delta w}{\delta y} & \frac{\delta w}{\delta z} \end{pmatrix} \left(\frac{\partial \mathbf{x}}{\partial \mathbf{u}} \right) . \tag{83}$$

and simplify (assuming that $x = x(\mathbf{u})$, $v = v(\mathbf{x})$, and $w = w(\mathbf{x})$) to get

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix} \left(\frac{\partial \mathbf{x}}{\partial \mathbf{u}} \right) . \tag{84}$$

And on taking the determinant of both sides of this we get (80).

PROBLEM:

Let $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. Find $\partial z/\partial x$ and $\partial z/\partial x$.

SOLUTION:

One way to solve this is to solve for z directly, but his would introduce messy square roots. So we'll use implicit differentiation as we have been doing. Let

$$F = x^2/a^2 + y^2/b^2 + z^2/c^2 - 1 \tag{85}$$

Now let $\boldsymbol{\eta} = (x, y)^t$, then

$$F(\boldsymbol{\eta}, (z(\boldsymbol{\eta})) = 0 . \tag{86}$$

On differentiating this we get

$$\frac{\delta F}{\delta \boldsymbol{\eta}} = \frac{\partial F}{\partial \boldsymbol{\eta}} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial \boldsymbol{\eta}} = 0 . \tag{87}$$

Solving this for $\partial z/\partial \boldsymbol{\eta}$, we get

$$\frac{\partial z}{\partial \boldsymbol{\eta}} = -\left(\frac{\partial F}{\partial z}\right)^{-1} \frac{\partial F}{\partial \boldsymbol{\eta}} . \tag{88}$$

From which we get

$$\left(\frac{\partial z}{\partial x} , \quad \frac{\partial z}{\partial y} \right) = -\left(\frac{2z}{c^2} \right)^{-1} \left(\frac{2x}{a^2} , \quad \frac{2y}{b^2} \right) = \left(-\frac{xc^2}{za^2} , \quad -\frac{yc^2}{zb^2} \right) . \tag{89}$$

PROBLEM:

Given

$$\begin{cases} xy + x^2u - vu^2 = 5 \\ x + 4uy - v^2u = 20 \end{cases} \tag{90}$$

find $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial x$,

SOLUTION: Let

$$\begin{cases} f_1 = xy + x^2u - vu^2 - 5 = 0 \\ f_2 = x + 4uy - v^2u - 20 = 0 \end{cases} \tag{91}$$

then

$$\begin{aligned}
\mathbf{u}_{,\mathbf{x}} &= -\mathbf{f}_{,\mathbf{u}}^{-1}\,\mathbf{f}_{,\mathbf{x}} \\
&= -\begin{pmatrix} x^2 - 2vu & -u^2 \\ 4y - v^2 & -2vu \end{pmatrix}\begin{pmatrix} y + 2xu & x \\ 1 & 4u \end{pmatrix} \\
&= -\frac{1}{D}\begin{pmatrix} -2vu & u^2 \\ v^2 - 4y & x^2 - 2vu \end{pmatrix}\begin{pmatrix} y + 2xu & x \\ 1 & 4u \end{pmatrix} \\
&= \frac{1}{D}\begin{pmatrix} 2vu & -u^2 \\ 4y - v^2 & -x^2 + 2vu \end{pmatrix}\begin{pmatrix} y + 2xu & x \\ 1 & 4u \end{pmatrix} \\
&= \frac{1}{D}\begin{pmatrix} 2vu(y + 2xu) - u^2 & 2xvu - 4u^3 \\ (4y - v^2)(y + 2xu) - x^2 + 2vu, & x(4y - v^2) + 4u(-x^2 + 2vu) \end{pmatrix}
\end{aligned} \tag{92}$$

where $D = 2vu(-x^2 + 2vu) + u^2(4y - v^2)$.

PROBLEM:

From Thomas ([3] p.542) we find the problem: The two equations

$$e^u \cos v - x = 0, \qquad e^u \sin v - y = 0 \tag{93}$$

define u and v as functions of x and y,

$$u = u(x, y)\,, \qquad v = v(x, y)\,. \tag{94}$$

Show that the angle between the two vectors

$$\frac{\partial u}{\partial x}\,\mathbf{i} + \frac{\partial u}{\partial y}\,\mathbf{j}\,, \qquad \frac{\partial v}{\partial x}\,\mathbf{i} + \frac{\partial v}{\partial y}\,\mathbf{j} \tag{95}$$

is a right angle (where $\mathbf{i}$ and $\mathbf{j}$ are orthogonal unit vectors).

SOLUTION: The solution is easy. Instead of introducing f_1 and f_2 as we did before, we can directly solve for x and y;

$$\begin{cases} x = e^u \cos v\,, \\ y = e^u \sin v\,, \end{cases} \tag{96}$$

Let $\mathbf{u} = \mathbf{u}(\mathbf{x}(\mathbf{u}))$ and differentiate this by $\mathbf{u}$, yielding

$$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \left(\frac{\partial \mathbf{x}}{\partial \mathbf{u}}\right)^{-1} . \tag{97}$$

Now

$$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \begin{pmatrix} e^u \cos v & -e^u \sin v \\ e^u \sin v & e^u \cos v \end{pmatrix}^{-1} = \begin{pmatrix} \cos v & \sin v \\ -\sin v & \cos v \end{pmatrix} . \tag{98}$$

Now, we'll show that the inner product of the two vectors is zero.

Now, in matrix form, the two vectors become $\dfrac{\partial u}{\partial \mathbf{x}}$ and $\dfrac{\partial v}{\partial \mathbf{x}}$, which are the first and second rows of (98). The inner product of these two vectors is

$$\begin{aligned} \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \begin{pmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{pmatrix} &= (\cos v,\, \sin v) \begin{pmatrix} -\sin v \\ \\ \cos v \end{pmatrix} \\ &= -\cos v \sin v + \cos v \sin v = 0 \,. \end{aligned} \tag{99}$$

Thus the two vectors are orthogonal, hence the angle between them is $\pi/2$.

Conclusion

SD has been presented here as a semantically advanced language for differentiation. It stresses that differential operators not only have formal properties, they also have distinct meanings that people have to deal with. SD obviates the need to make regular use of such cumbersome terms as "total" partial derivatives and "partial" partial derivatives, as I have seen in some references.

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