Unsolved Problem Solved by Structured Differentiation

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Abstract

Structured Differentiation (SD) is used to solve a problem I found lingering unsolved on a physics forum since October 2012.

1 Introduction

> Matters of notation play a considerable role in connection with the chain rule. Wide varieties of usage exist in mathematical writing where the chain rule is concerned.
>
> — Taylor & Mann [1]

I recently stumbled across the following question found at[1]

`https://www.physicsforums.com/threads/hard-partial-derivatives -question.646666/`

The problem statement, all variables and given/known data:
Taking k and ω to be constant, [find] $\partial z/\partial\theta$ and $\partial z/\partial\phi$ in terms of x and t for the following function

$$z = \cos\left(kx - \omega t\right), \tag{1}$$

where $\theta = t^2 - x$ and $\phi = x^2 + t$.

Note 1: I make the reasonable assumption that x and t are the independent variables of the problem.

Note 2: In SD a partial derivative is an explicit derivative.

[1] The time of the first presentation of this problem's solution by SD is April, 2019.

2 What this problem is not

This problem is *not* (easily) solvable by a simple algebraic maneuver in which we first solve for x, t in terms of θ, ϕ. Why? Because the algebra involved in doing so is tedious and the resulting functions are difficult to differentiate.

Specifically, when I entered the string

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solve for x,t in \theta=t^2-x, \phi = x^2+t
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into WolframAlpha, which correctly interpreted the command and returned the variables x, t in terms of θ, ϕ, I got back a nightmare of complicated functions to deal with. (Try it yourself and see.)

However, if the original problem had been algebraically simpler, say, $\theta = t - x$, $\phi = x + t$, we could easily solve for x, t to get

$$x = \frac{\phi - \theta}{2}\,, \quad t = \frac{\phi + \theta}{2} \tag{2}$$

In which case, (1) would become

$$z = \cos\big(k\frac{\phi - \theta}{2} - \omega\frac{\phi + \theta}{2}\big) = \cos\big(\frac{k - \omega}{2}\phi - \frac{k + \omega}{2}\theta)\big)\,. \tag{3}$$

Then, finding $\partial z/\partial\theta$ and $\partial z/\partial\phi$ in terms of x and t would be straightforward.

However, what can't be accomplished by algebra can sometimes be accomplished by differentiation, which, I think, is the point of this problem.

3 Solution 1.

In SD, we write down a functional dependence and then take a total derivative:

Let

$$\mathbf{u} = \begin{bmatrix}\theta\\ \phi\end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix}x\\ t\end{bmatrix} \tag{4}$$

with

$$z = z(\mathbf{u}(\mathbf{x}))\,. \tag{5}$$

Now, we take a total derivative across (5) by $\mathbf{x}$ and then use the chain rule:

$$\frac{\delta z}{\delta\mathbf{x}} = \frac{\partial z}{\partial\mathbf{u}}\frac{\delta\mathbf{u}}{\delta\mathbf{x}}\,. \tag{6}$$

Expanding this, we get

$$\left[\frac{\delta z}{\delta x}, \frac{\delta z}{\delta t}\right] = \left[\frac{\partial z}{\partial\theta}, \frac{\partial z}{\partial\phi}\right]\begin{bmatrix}\frac{\delta\theta}{\delta x} & \frac{\delta\theta}{\delta t}\\ \frac{\delta\phi}{\delta x} & \frac{\delta\phi}{\delta t}\end{bmatrix}\,. \tag{7}$$

Next, some simplifications. First, given the functional form of z on independent variables x and t in (1), the total derivatives of z by x and t can be replaced by explicit derivatives. By similar reasoning, the total derivatives of θ and ϕ by x and t can also be replaced by explicit derivatives, yielding

$$\left[\frac{\partial z}{\partial x}, \frac{\partial z}{\partial t}\right] = \left[\frac{\partial z}{\partial \theta}, \frac{\partial z}{\partial \phi}\right] \begin{bmatrix} \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial t} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial t} \end{bmatrix} . \tag{8}$$

At this point, we can solve (8) for the derivatives we want

$$\left[\frac{\partial z}{\partial \theta}, \frac{\partial z}{\partial \phi}\right] = \left[\frac{\partial z}{\partial x}, \frac{\partial z}{\partial t}\right] \begin{bmatrix} \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial t} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial t} \end{bmatrix}^{-1} , \tag{9}$$

wherever the inverse matrix exists. Now, all the derivatives on the RHS are easy to perform, yielding

$$\begin{aligned} \left[\frac{\partial z}{\partial \theta}, \frac{\partial z}{\partial \phi}\right] &= \Big[-k\sin(kx-\omega t),\ \omega \sin(kx-\omega t)\Big] \begin{bmatrix} -1 & 2t \\ 2x & 1 \end{bmatrix}^{-1} \\ &= \Big[-k\sin(kx-\omega t),\ \omega \sin(kx-\omega t)\Big] \frac{1}{-1-4xt} \begin{bmatrix} 1 & -2t \\ -2x & -1 \end{bmatrix} \\ &= \Big[-k\sin(kx-\omega t),\ \omega \sin(kx-\omega t)\Big] \frac{1}{1+4xt} \begin{bmatrix} -1 & 2t \\ 2x & 1 \end{bmatrix} . \end{aligned} \tag{10}$$

Hence, the derivatives we seek are

$$\begin{aligned} \frac{\partial z}{\partial \theta} &= \sin(kx-\omega t)\frac{k+2\omega x}{1+4xt} , \\ \frac{\partial z}{\partial \phi} &= \sin(kx-\omega t)\frac{-2kt+\omega}{1+4xt} . \end{aligned} \tag{11}$$

By the way, to have Wolframalpha calculate the matrix inverse, one could enter

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find inverse [[-1, 2t], [2x, 1]]
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And to calculate the two partial derivatives, enter

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[-k\sin(kx - \omega t),\omega\sin(kx-\omega t)] (inverse [[-1, 2t],[2x, 1]])
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4 Solution 2.

It's an interesting fact that Equation (5) is not the only way to parameterize z's dependency on $\mathbf{x}$ and $\mathbf{u}$. We could also do it this way

$$z = z(\mathbf{x}(\mathbf{u})) . \tag{12}$$

As before, we take a total derivative — this time by $\mathbf{u}$ — and then use the chain rule (with simplifications)

$$\frac{\partial z}{\partial \mathbf{u}} = \frac{\partial z}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \mathbf{u}} \,. \tag{13}$$

Expanding this, we get

$$\left[\frac{\partial z}{\partial \theta}, \frac{\partial z}{\partial \phi}\right] = \left[\frac{\partial z}{\partial x}, \frac{\partial z}{\partial t}\right] \begin{bmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial t}{\partial \theta} & \frac{\partial t}{\partial \phi} \end{bmatrix} \,. \tag{14}$$

So, at first look, it seems that this approach is the more direct route to solving for the derivatives we seek. But this is not as straightforward as it seems, because we still don't know how to directly perform the partial derivatives in the square matrix. But there is a trick we can use: We can convolute $\mathbf{x}$ through $\mathbf{u}$ to get[2]

$$\mathbf{x} = \mathbf{x}(\mathbf{u}(\mathbf{x})) \,. \tag{15}$$

On differentiating this by $\mathbf{x}$, we get

$$\frac{\delta \mathbf{x}}{\delta \mathbf{x}} = \frac{\partial \mathbf{x}}{\partial \mathbf{u}} \frac{\delta \mathbf{u}}{\delta \mathbf{x}} \,. \tag{16}$$

But since the components of $\mathbf{x}$ are functionally independent of each other, $\frac{\delta \mathbf{x}}{\delta \mathbf{x}}$ is the identity matrix $\mathbf{I}$. And making the substitution $\frac{\delta \mathbf{u}}{\delta \mathbf{x}} \to \frac{\partial \mathbf{u}}{\partial \mathbf{x}}$, we get

$$\mathbf{I} = \frac{\partial \mathbf{x}}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \,. \tag{17}$$

Therefore,

$$\frac{\partial \mathbf{x}}{\partial \mathbf{u}} = \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^{-1} \,. \tag{18}$$

Now, substituting this into (13), we get

$$\frac{\partial z}{\partial \mathbf{u}} = \frac{\partial z}{\partial \mathbf{x}} \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^{-1} \,, \tag{19}$$

which, in component form, is just (9).

References

[1] A. Taylor and W.R. Mann. *Advanced Calculus,* 2ed. John Wiley & Sons. New York (1972).

[2]In SD, a variable is said to be *convoluted* if it is functionally dependent on itself through at least one other variable. The meaning here is different than its use in Laplace transform theory.