My Defence of Structured Differentiation from 1999, 2

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Abstract

Here I review my defence of Structured Differentiation which I had made in 1999 on sci.math.

1 Introduction

In 1999, I made a defence on sci.math of my notation in Structured Differentiation (SD), which is a notational system I invented to deal with the many confusing (and well-recognized) features that commonly arise in multi-variable calculus. A mathematician on the newsgroup thought he should counter my claims and I'll present his arguments, and my counterarguments. The reader can decide the merits of my system for him or herself.

I think it will become obvious to the reader that the reason partial derivatives is a confusing subject is simply because it employs too few symbols to chase too many concepts. All SD does is to add in a couple more symbols to better distribute the cognitive workload.

Of all the mathematics subjects I've published on in the AJNP the most controversial one is what I call *Structured Differentiation* (SD), which reorganizes and reformulates the so-called theory of "partial differentiation." "Defender" is an alias for a mathematician that defended the status quo for doing so-called partial differentiation [as it was commonly accepted at that time] against my presentation of SD (I have interjected "editorial" comments within square brackets.):

2 Defender's reply (17 November)

Subject: Re: partial derivation
Date: 17 Nov 199915:08:10
From: Defender
Newsgroups: sci.math
In article <38327732.984435FD> Patrick Reany writes:
> [snip]

> I trust that my example problems below will prove that SD is
> superior in dealing with real problems taking derivatives of
> functions.

"Real problems" which show up as third-semester calculus questions, perhaps. (Granted these problems often arise later on as well, but there are certainly more difficult problems which are "real".)
> [snip]
> The total derivative is the ONLY differential operator that can be I
> applied across an equal sign [...]

This is complete nonsense. Rule Number One of Algebra (if you have to number the rules) is, what you do to one side of the equation, you I must do to the other, and if an equation is valid, then if you can do something to one side, then you can to the other side as well. Your statement above suggests to me that you don't really understand functions yet (which would fit nicely with my claim that problems arise not from partial but from misunderstanding of functions).

[Defender had made up his mind about the controvery before ever giving my position a fair hearing.]

> [snip]
> (BTW, the ordered set of independent variables is referred to as the
> 'fundamental' in SD. One reason for this is because it is much
> easier in change-of-variable problems to think of having a new
> fundamental dependent on the old fundamental than to have a new set
> of independent variables dependent on a set of old independent
> variables.)

All you're doing here is replacing the universally accepted phrase "independent variables" with the word "fundamental".

[Actually, I just gave a GOOD reason to advance an alternative name. In particular, when I studied thermodynamics, I had to comprehend how the 'independent variables' could change in a given problem. We start off with the independent variables being V and T and then have to make them P and T, and then S and V and so forth. I kept having the psychological difficulty with the notion that if a variable is 'independent' then it can't become un-independent. Looking back, this psychological stumbling block seems silly now, but back then, it was very real. I introduced the term 'fundamental' to overcome the psychological resistence I had, changing it for a different set of 'fundamental' variables, which seemed easier to comprehend.]

> [snip]
> I was first made aware of this 'obvious' concept, not from my
> calculus books, but from a book on statistical mechanics by Kerson
> Huang.

Thermodynamics and statistical mechanics books are far from the top of my list of books from which to get mathematical insights. (As I said earlier, things made sense except for their use in thermodynamics. All the thermo books I have seen [admittedly not many] exhibit a great mis- or lack-of-understanding of mathematics.)

> [snip]
> More common with mathematicians is to 'reduce to primitive form,' as
> I call it.

Most people call it "technically accurate notation".

> That is to replace a function having implicit dependence on a
> variable with a function that has only explicit dependence on the
> variable. For example, the equation $W = W(t, x(t))$ would have the
> function W reduced to primitive form by writing $f(t) = W(t, x(t))$,
> say.

The equation "$W = W(t, x)$" is technically horrible (in this context). If you have problems computing derivatives, you would be well advised to introduce the new function $f(t) = W(t, x(t))$.

[SD has NO problems at all with either form!]

> [snip]
> In cases where in physics the implicit vs explicit dependence of a
> variable have very different meaning and physical origins (such as
> in the case of the Boltzmann transport equation above and also for
> Lagrangians); such equations do not like being contorted into
> primitive form which, though allowable formally, would change the
> meaning of the equations.

Many times I have converted transport-like equations (or similar things) into technically accurate notation. The meaning was not changed, but it was certainly clarified, to great benefit both conceptually and computationally. That you apparently are unable to do so without "changing the meaning of the equation" suggests to me (again) that your problems are with understanding functions, not with computing partial derivatives.

[What Defender apparently does not appreciate is how physicists treat explicit and implicit functional dependence. This distinction may seem arbitrary to a mathematician but not to a typical physicist. In any case, both mathematicians and physicists effectively use the explicit/implicit duality to good advantage in many kinds of problems. In fact, a common strategy in applied mathematics involving total derivatives is to find formulas that use **only** explicit derivatives because those are truly unambiguous derivatives to perform. For an example of this, see the change-of-variable problem from x-y coordinates to polar coordinates found in my many papers on SD on my website.]

> Let me demonstrate just how clean and minimalist SD is on a real problem.

Here's my proof of the statement (taken as far as you took your proof): Let $x = (x_1, x_2)$, $G = (G_1, G_2)$, $g = (g_1, g_2)$. Define $h(x) = (x, f(x))$. Now write all these row-vectors as column-vectors. Note that $g(x) = G(h(x))$. Apply the chain rule (use a fixed-width font to view this): $Dg(x) = DG(h(x))$

$$Dh(x) = \begin{bmatrix} D1G1 & D2G1 & D3G1 \\ D1G2 & D2G2 & D3G2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ D1f & D2f \end{bmatrix} \tag{1}$$

Expand and regroup. (The notation $D3G2$ means $\partial G_2/\partial z$, for example.)

[In the following papers, I will adopt the full benefit of LaTeX to render all the mathematics.]

> Now I will address the accusation that the copartial is not well-
> defined in SD. I will demonstrate that it is well defined in a
> special case and trust that it is obvious how to generalize.

I know how you are defining it, and I know how to generalize your definition. I also know that it is not well-defined, in the same sense in which you are saying that "∂_t" in the standard notation is not well-defined. Here is an example of why it is not well-defined. Compute the copartial wrt t of $f(t, x) = t + x$. I get $x'(t)$. [So do I.] Compute the copartial wrt t of $g(t, y) = 2t + y$. I get $y'(t)$. [So do I.] Now, suppose $x(t) = 2t$, and suppose $y(t) = t$. Then $f(t, x) = g(t, y)$, but in the first case, the value of the copartial is 2, while in the second, the value is 1. This is *exactly* the same issue which arises in the standard notation, namely, ∂_x is not defined until you have defined your coordinate system. Your notation just buries the issue a layer deeper, which will cause more confusion when it does arise.

[All I can say at this point is that Defender has misinterpreted the copartial derivative as ill-defined, when the real problem is that it is an incomplete operator. The rest of this paper is my reply to this criticism.]

To derive a correct result by differentiation on the equation

$$f(t,x) = g(t,y)\,, \tag{2}$$

(which we assume is true on some domain), we should apply the following recipe:

1. Take a total (deltal) derivative across the equation.
2. Take the parametric split on the deltal derivatve.
3. Evaluate and simplify.

So, this is what Defender gave us:

$$f(t,x) = t + x\,, \qquad g(t,y) = 2t + y\,, \tag{3}$$

$$\frac{\not\partial f}{\not\partial t} = x'(t)\,, \qquad \frac{\not\partial g}{\not\partial t} = y'(t)\,. \tag{4}$$

But we are also given that

$$x(t) = 2t\,, \qquad y(t) = t\,, \tag{5}$$

$$x'(t) = 2\,, \qquad y'(t) = 1\,. \tag{6}$$

So, if we apply the copatial derivative to both sides of (2), we get

$$\frac{\not\partial f}{\not\partial t} = \frac{\not\partial g}{\not\partial t}\,, \tag{7}$$

we get that $2 = 1$, which is false. But this problem is a perfect example of why we **shouldn't** apply either the partial (explicit) or copartial (implicit) derivatives on an equation by themselves. Each is by itself an incomplete differential operator. Only by adding them together do we get a complete differential operator. Applying only one of these might work in cases, but will fail in others.

But what happens when we follow the rules of SD to Eq. (2)? We get

$$\begin{aligned} \frac{\delta f}{\delta t} &= \frac{\partial f}{\partial t} + \frac{\not\partial f}{\not\partial t} & \Big| \quad \frac{\delta g}{\delta t} &= \frac{\partial g}{\partial t} + \frac{\not\partial g}{\not\partial t} \\ &= 1 + \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} & \Big| \quad &= 2 + \frac{\partial g}{\partial y}\frac{\partial y}{\partial t} \\ &= 1 + (1)(2) = 3 & \Big| \quad &= 2 + (1)(1) = 3\,. \end{aligned} \tag{8}$$