My Defence of Structured Differentiation from 1999, 5

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Abstract

Here I review my defence of Structured Differentiation which I had made in 1999 on sci.math.

1 Introduction

In 1999, I made a defence on sci.math of my notation in Structured Differentiation (SD), which is a notational system I invented to deal with the many confusing (and well-recognized) features that commonly arise in multi-variable calculus. A mathematician on the newsgroup thought he should counter my claims and I'll present his arguments, and my counterarguments. The reader can decide the merits of my system for him or herself.

I think it will become obvious to the reader that the reason partial derivatives is a confusing subject is simply because it employs too few symbols to chase too many concepts. All SD does is to add in a couple more symbols to better distribute the cognitive workload.

Of all the mathematics subjects I've published on in the *AJNP*[1] the most controversial one is what I call *Structured Differentiation* (SD), which reorganizes and reformulates the so-called theory of "partial differentiation." "Defender" is an alias for a mathematician that defended the status quo for doing so-called partial differentiation [as it was commonly accepted at that time] against my presentation of SD (I have interjected "editorial" comments within square brackets.):

2 My Reply (18 November)

Subject: Re: partial derivation
Date: Thu, 18 Nov 199923:43:58 -0700
From: Patrick Reany
Newsgroups: sci.math

Defender feels that we have said everything of interest about the subject and, although I don't agree, I will expect no further replies from him in this thread. Anyway, thank you, Defender, for a very good defense of the conventional formalism of partial differentiation. It was an instructive debate.

I have been arguing for SD since 1982. I have had many debates. I have had many converts and many detractors. But how is that possible? This is supposed to be mathematics – an objective endeavor. How can people disagree so strongly on such a fundamental aspect of mathematics? There's more at stake here than just confusing notation. What is really at stake here is mathematical integrity. No other area of mathematics allows us to define our terms and then immediately allow us to ignore those definitions. Why should partial differentiation be an exception?

[1] The *Arizona Journal of Natural Philosphy* is a defunct journal I published long ago.

Defender wrote: >r
>> [snip] ...Since you say you have no difficulty *using* ‘partial
>> derivatives,’ suppose you define what one is for us.
>
> Let $f : R^n \to R$ be a function of the n variables $(x_1 \ldots, x_n)$. The
> partial derivative of f wrt x_1 (if it exists) at the point $(x_1, \ldots, x_n)$
> is

$$\lim_{h \to 0} \frac{f(x_1 + h, x_2, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{h}. \tag{1}$$

> I’ll denote this as $D_1 f(x)$. (Of course this presupposes understanding
> of the limit.)

Here Defender gives the definition most often given for the ‘partial derivative.’ Note that it is an explicit derivative w/r x_1, i.e., it does not track the variation of the function, f, through any variable except through the explicit dependence f has on x_1. So, even if f should vary by x_1 through x_2, say, this ‘partial’ derivative will not take account of that. We need another derivative for that, but Defender doesn’t define one. Neither does convention, really. All I ever seem to find are ad hoc attempts to get something like a total derivative out there, but it remains elusive.
[snip]
> And while we’re in the spirit of presenting logical justifications of the
>> processes of differentiation, what differential operator did you
>> apply to both sides of
>>
>> $g(x) = G(h(x))$
>>
>> to get
>>
>> $Dg(x) = DG(h(x))Dh(x)$?
>
> This is a direct quote of part of the statement of the chain rule.
> Theorem: If h is differentiable, G is differentiable, and the
> composition $(G \circ h)$ (defined by $(G \circ h)(x) = G(h(x))$) is
> defined, then it is differentiable, and
> $D(G \circ h)(x) = DG(h(x))Dh(x)$.
>

> Given the definition of g, namely, $g(x) = G(h(x))$, it is immediate that
> g is differentiable, with its derivative given by the above formula.
>
>> This is one of my biggest irritations against conventional ‘partial
>> differentiation’ legerdomain. Either the operator is never
>> specifically identified or it’s identified as a ‘ from Defenderpartial derivative’
>> which is supposed to be interpreted in context.
>
> “The Operator” which I applied was the derivative. I then expressed
> the various derivatives in terms of the matrices as represented in the
> standard coordinate systems on R^2 and R^3.

What Defender is saying is that conventional partial differentiation has some invisible operator, called ‘the derivative,’ which has been applied, out of sight of the beginner, and then we rely on the result of a theorem to write the derivative. Oh, there are so many questions I’d still like to

ask Defender, like: What symbol represents this invisible 'derivative'? And what is its definition? And how does it relate to the total derivative of ordinary differential calculus when there is onlyone independent variable? How does it relate to our explicit derivative defined above? SD has good answers to all these questions.
[snip]

> I would *very* much prefer that the learner learn and understand
> things such as functional notation, composition of functions, etc.,
> rather than learning formalism (without theory) which (perhaps) aids
> computation in some set of problems.

[A partial consession from Defender on the usefulness of SD? (No pun intended.) As for the theory, it resides in my many papers on SD.]

I have written a great deal more on total differentiation than I'll offer as a post. My immediate goal was to warn against the confusions and mathematically incorrect procedures used in partial differentiation. But of course there's more to the subject than that. As I stated before, the real intrinsic difficulties are the complex functional dependencies (which Defender refers to) that variables and function can mutually have. But it was impossible for me to even begin to understand that aspect until I had the right understanding of what a differential operator is in the subject and how to break it into explicit and implicit parts, and how it relates to the total derivative of ordinary differential equations. It is only then that the student has a fair chance to understand the subject and not just learn to apply theorems without understanding.

>>> Compute the copartial wrt t of $f(t,x) = t + x$. I get $x'(t)$.
>>
>> In SD:
>> $\not\partial_t f(t,x) = \partial_x f(t,x)\delta_t x$
>> $= \partial_x (t+x)\delta_t x$
>> $= \delta_t x$
>>
>> and this is as far as I am allowed to go. To say that $\delta_t x$ is
>> equal to $x'(t)$ requires you to make some tacit assumption about the
>> functional dependence of x on t, which is unjustified by the
>> information you gave. Are you assuming that $x = x(t)$? No, you can't
>> because you don't like that notation.
> > OK, you caught me. (I said it was bad, I didn't say I never do it.)
> :-)

[Defender fell into this trap of his own devising!]

>
> Let me rephrase: suppose $x = X(t)$. Compute the copartial of $f(t,x) = t + x$.
> Then the answer is $X'(t)$. Similarly, $y = Y(t)$; the copartial of
> $g(t,y) = 2t + y$ is $y'(t)$. Now let $X(t) = 2t$, $y(t) = t$, and evaluate the above
> copartials; the answers are respectively 2 and 1. However,
> $f(t,x) = g(t,y)$, and this discrepancy (2 vs 1) is what I mean by the
> copartial not being well-defined; more precisely, it is not
> invariantly defined. The exact same statement is true of the partial
> derivative: its value depends on the coordinate system.

Yes, it is true of the partial (explicit) derivative. But how committed are you to maintaining that a partial derivative is an explicit derivative? My guess is that your notion of "invariantly defined" and mine of "complete equational operator" is the same thing. An operator is said to be "complete equational" if, when applied to any equation, it *always* maintains the equality. As I said many, many times before, the total derivative has this property, the explicit and implicit derivatives do not. Thus the total derivative is the central operator of so-called "partial differentiation" just as it is in ordinary differentiation! In fact SD makes it obvious that the two subjectsare just the same subject, and distinguished only by the number of independent variables about.

[It's too bad that Defender never really gave my argument a chance, for then he'd have seen what I was getting at much sooner in the thread and we could have resolved this whole thread in 1/5th the time. But as it is, I find that I have to keep re-stating myself to finally get any point across to Defender.]

> [snip]
>> You aren't allowed to apply an implicit derivative operator to two
>> sides of an equation! In my first post I showed how disobeying this
>> central rule can get you into trouble when differentiating $F(x, y, z)$
>> = 0 explicitly by x, say, getting wrongly that $\partial_x F = 0$.
>> Remember?
>
> Yes, I remember. I remember thinking "blatant misunderstanding or
> intentional deception".

Avoiding misunderstandings was the whole point of my first post in this thread. Back in 1982, I was definitely experiencing blatant misunderstanding of so-called partial differentiation!

> If $F(x, y, z) = 0$ for all x, y, z (in a neighbourhood of a point), then
> $\partial_x F = 0$. On the other hand, if you are using $F(x, y, z) = 0$ to define
> a set of points (namely, the set of points which satisfy the
> equation), then you must recognize that the equation is true only at
> certain values (x, y, z). Generically, the equation will be false, and
> if you want to apply the "partial derivative operator" ∂_x to
> the equation, you will get something which is generically false.

That was my point too. The 'del' (explicit) derivative is an incomplete equational operator [i..e, 'del' = ∂]. Although you can always apply it to an expression, you can't always apply it both sides of an equation and get back an equation. [This defines what I meant by an ' incomplete equational operator'.]

> What you can do, though, is appeal to the implicit function theorem,
> which says (in part, and given additional hypotheses) that the local
> solution of the equation $F(x, y, z) = 0$ is given by $z = g(x, y)$. Then, by
> construction, $F(x, y, g(x, y)) = 0$ will be true for all x, y (in a
> neighbourhood), and then applying the "partial derivative operator"
> ∂_x to this always-true equation will give another true
> equation: $D_1F(x, y, g(x, y)) + D_2F(x, y, g(x, y))D_1g(x, y) = 0$.

No. This is the same old error done here that convention practically forces us to make and that most people make, I presume. We have already defined above the "partial derivative operator" as an explicit derivative and we cannot apply it across an equal sign and guarantee that the equality will hold. In this case it certainly doesn't hold. Aren't mathematicians supposed to hold true to their definitions?

In SD this ghostly derivative' that Defender refers to comes out of the shadows and shows itself as a deltal derivative – a total derivative. The term $D_1F(x, y, g(x, y))$ is an explicit derivative – a partial derivative by our previous definition above. (It's important not to get confused about having two words apply to one derivative. The term ‘partial’ is the name of the derivative, while the term ‘explicit’ is the meaning of the derivative. The greek ‘del’ sign is used as the symbol denoting the partial derivative in SD.)

So the equation

$$D_1 F(x, y, g(x, y)) + D_3 F(x, y, g(x, y)) D_1 g(x, y) = 0 \tag{2}$$

is the result of applying the (total) deltal derivative to the equation $F(x, y, g(x, y)) = 0$. However, SD prefers the derivatives to take the following forms:

$$\frac{\delta}{\delta x} F(x, y, z(x, y)) = \frac{\delta}{\delta x} 0 = 0 \tag{3}$$

which expands to

$$\frac{\partial}{\partial x} F(x, y, z(x, y)) + \frac{\partial}{\partial y} F(x, y, z(x, y)) \frac{\delta y}{\delta x} + \frac{\partial}{\partial z} F(x, y, z(x, y)) \frac{\delta}{\delta x} z(x, y) = 0\,, \tag{4}$$

which reduces to

$$\frac{\partial}{\partial x} F(x, y, z(x, y)) + \frac{\partial}{\partial z} F(x, y, z(x, y)) \frac{\partial}{\partial x} z(x, y) = 0 \tag{5}$$

on assumption that x and y are fundamental (independent) variables. [Or, at the very least, that y is independent of x, hence $\frac{\delta y}{\delta x} = 0$, and thus $\frac{\delta z}{\delta x} = \frac{\partial z}{\partial x}$.]

[snip]

> As this discussion seems to be going nowhere, I’m happy to drop it if you
> are.
>
> Defender.

Thanks for accepting my challenge to the establishment and defending it so well. I found the debate to be interesting.

cheers,

Patrick