My Defence of Structured Differentiation from 1999, 0: Series Introduction

P. Reany

February 7, 2024

A clue is anything that doesn't happen they way it oughtta happen.
– Harry Orwell (David Janssen)
[a character from *Harry O*, S1E1 (1974)]

1 Series Introduction

In mid-November 1999, I made a defence on sci.math of my notation in Structured Differentiation (SD), which is a notational system I invented to deal with the many confusing (and well-recognized) features that commonly arise in multi-variable calculus. A mathematician (whom I named 'Defender') on the newsgroup thought he should counter my claims and I'll present his arguments, and my counterarguments. What follows this introduction to the series is a set of eight back-and-forths between us. The reader can decide the merits of my system for him or herself.

In have been studying math for a long time. By 1982, I had become frustrated with the subject known as partial differentiation. But the more I tried to understand this subject by reading from evermore sources, the more confused I became. I had to endure ghost derivatives that were talked about and used, but never named nor given a representation on paper. Over and over, authors presented partial differentiation in ways that defied my expectations, leaving me feeling (as Harry Orwell put it): that things are happening in ways other than they ought to happen. By that year, I concluded that the subject is dysfunctional. I just couldn't understand crucial bits of it.

I suppose that if I had know at that time the wisdom of John von Neumann, I might have given up the search for a deep understanding.

Young man, in mathematics, you don't understand things. You just get used to them.
– John von Neumann

Then again, maybe not. Anyway, I decided to spend whatever time it took until I would get to understand 'partial derivatives'. I studied from many advanced calculus books and physics textbooks. Their wide use of conflicting standards and terminology is confusing.

In one form or another, the total derivative of ordinary differential calculus will get generalized to the following form:

$$\left(\frac{\partial}{\partial t}\right)_{\text{total}} = \left(\frac{\partial}{\partial t}\right)_{\text{explicit}} + \left(\frac{\partial}{\partial x}\right)_{\text{implicit}} . \tag{1}$$

So, in eight posts, I defended my Structured Differentiation (SD) and Defender defended the status quo. But to understand SD, the reader needs to know some of the definitions and notations I use within SD.

2 Notations and Definitions

In SD, the set of variables on which a function is explicitly dependent are called the *variants* of the function. For example, in the function $F(x,t,u)$, F has variants x,t,u. A function whose variants are all mutually independent of each other is said to be *primitive*.

In SD the generalized total derivative is $\dfrac{\delta}{\delta x}$. This derivative can always be split into the sum of an explicit and an implicit part, where the explicit derivative is given by $\dfrac{\partial}{\partial x}$ and the implicit derivative is $\dfrac{\not\partial}{\not\partial x}$, called, respectively, the 'partial' and 'copartial' derivative. When written out, we get the following:

$$\frac{\delta}{\delta x} = \frac{\partial}{\partial x} + \frac{\not\partial}{\not\partial x}\,. \tag{2}$$

I call this split a "parametric split," because which of these terms is zero, if either of them is, depends on how one chooses to parameterize a given fiunction F.

For example, consider the two functions F,G given by

$$F(y,z) = y^2 + z(x,y)\,, \quad G(x,u) = x + u\,, \tag{3}$$

where y is independent of x and G is primitive. Then $\dfrac{\partial F}{\partial x} \equiv 0$ and $\dfrac{\not\partial F}{\not\partial x} = \dfrac{\partial F}{\partial z}\dfrac{\partial z}{\partial x}$. And for G: $\dfrac{\partial G}{\partial x}$ is not identically zero, but its implicit derivative by x is identically zero, or

$$\frac{\not\partial G}{\not\partial x} \equiv 0\,. \tag{4}$$

If we apply the deltal derivative by x, say, to a primitive function, say $F(x,y,z)$, the copartial derivative is identically zero. Therefore, we get that

$$\frac{\delta F}{\delta x} = \frac{\partial F}{\partial x}\,. \tag{5}$$

This presents to us two equivalent interpretations: a) The partial derivative of standard mathematics is a generalized total derivative, and b) The total derivative of a primitive function is an explicit derivative.

SD follows the long tradition used within physics of **not** introducing new symbols to distinguish functions from their variable representations. In mathematics, typically, the function $f(x,y)$ might be represented by a variable, say z, to get

$$z = f(x,y)\,. \tag{6}$$

In abstract mathematics, this doesn't conflict with any standards because all variable and function letters are free to use. (At this moment, I can't think of any reserved letters in mathematics except i, the unit imaginary, and a few symbols used in statistics, but those reserved symbols are also reserved in physics.)[1] But not so in physics, in which (typically) s is distance and S is entropy, and F is force, v is speed, and V is volume, and a is acceleration, A is the vector potential, and B is the magnetic field, and c is the speed of light, and on and on. There are so many standardized variables and functions in physics that it's hard to barrow one on the fly to replace a function with a different symbol for its variable. So, to lessen the burden of overuse, for example, for the function/variable z

$$z = z(x(t))\,, \tag{7}$$

[1]Oh yes, some greek and hebrew letters are reserved in set theory. And that weird theta for the theta functions, and that weird P for the Weierstrass elliptic functions. ;-)

using the chain rule to differentiate by t (in typical physics style), we get

$$\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt}\,; \tag{8}$$

whereas in mathematics, one might feel it's more correct to do the following.

$$z = f(g(t))\,, \quad x = g(t)\,, \tag{9}$$

to get

$$\frac{df}{dt} = \frac{df}{dx}\frac{dg}{dt}\,. \tag{10}$$

Thus we see how the mathematics community and the physics community have gone on different paths in terms of their customs regarding differentiation. The value of SD is not to find fault with one or the other, but to find some common way to think about what is going on in each of them.

The purposes of SD are: 1) To find a reasonable means to generalize ordinary differentiation to partial differentiation. 2) To remove as much ambiguity from the subject as possible. 3) To add to the ether enough terminology and symbology to clarify the subject, and provide a minimal vocabulary for understanding and communicating the concepts of differentiation. 4) To provide a learner's language that students can learn the essentials of advanced differentiation, which can then be used to understand how the various systems of differentiation are used.

About the time I was deeply involved in the initial development of SD, I was also taking my first official computer science class (1982), which was Pascal. Pascal never became a big enterprise programming language, but it was considered useful as a teaching language to get across progamming concepts and skills. Likewise, at least for me, SD is the 'Pascal' of advanced differentiation.

3 The Partial Derivative

Later in the series of posts, Defender will provide us with this definitiion of the partial derivative:

Let $f : R^n \to R$ be a function of the n variables $(x_1 \ldots, x_n)$. The partial derivative of f wrt x_1 (if it exists) at the point $(x_1, \ldots, x_n)$ is

$$\frac{\partial f}{\partial x} \equiv \lim_{h\to 0} \frac{f(x_1 + h, x_2, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{h}\,. \tag{11}$$

when the limit exists. The main issue here is that the function f is primitive, that is, its variants are mutually independent. The trick is to decide how to generalize this partial derivative to apply to functions that are not primitive. This single point is one of the key points of whimsical variations on notations and a profound source of contention between my SD and many conventional systems of differentitation.

On

`https://math.libretexts.org/Bookshelves/Calculus/Map%3A_University_Calculus_(Hass_et_al)`
`/13%3A_Partial_Derivatives/13.3%3A_Partial_Derivatives`

under the Partial Derivatives title, we are told that:

"The idea to keep in mind when calculating partial derivatives is to treat all independent variables, other than the variable with respect to which we are differentiating, as constants."

Although this statement is true, it's also very misleading. Of course, the derivative of an independent variable with respect to some other independent variable is zero because it must be! In fact, that is the key to how you would define two variables, such as x and y, say, to be independent of each other:

$$\frac{\delta x}{\delta y} \equiv 0\,, \quad \frac{\delta y}{\delta x} \equiv 0\,. \tag{12}$$

Nevertheless, I clearly remember being given this kind of characterization of 'partial differentiation'. If one takes the derivative of a primitive function by one of its variants (i.e., a partial derivative), then the derivative can only be explicit and cannot have any implicit dependence on that variant.

4 The Psychological Factors for Resistance to SD

Those who went through the learning curve for partial differentiation will like come out on the other side as belonging to one of the following groups: A) Those that have acclimated to it and no longer see it as' problematic', and B) those that retain a sense that the subject is rife with confusion and inconsistencies. So that those in the latter group will likely have a sympathetic reception to SD, while those of the former group are likely to dismiss it out of hand, and then deride it the devil's work.

What makes this latter reception of some consternation to me is that I can show from the authors of the advanced calculus books themselves that all is not well in the notational aspects of standard partial differentiation. And when I point this out to those people, they seem to ignore it or even go so far as to attempt to gaslight me that those authors couldn't have really meant that. Well, they did!

5 Sources

The eight posts of this series were originally written in plain text. Then, long ago I converted them into HTML files. This last transcription of the posts converts them into LaTeX. I hope that I have not degraded the texts in any significant way in the two transcriptions. One thing I am sure of, I don't want to go back to the sci.math sources themselves. So what is here will have to do.

6 Afterword

Keep an open mind – that's the secret! (Doctor Who)

The two terms that I invented for SD that seem to be quite useful within any discussion of partial differentiation are 'variant' and 'primitive'. With them, one can say something like, "Given the following equation

$$g(x, y) = G(x, y, z(x, y))\,, \tag{13}$$

we have reduced G to its primitive form with respect to its variants." In fact, this kind of reduction is so common in standard mathematics, that I don't see how mathematicians get along without them, or something like them.

I wrote this series introduction within the last week, that being 24 years after I responded to Defender, and being 42 years after my first meager attempts to formulate Structured Differentiation.

You may notice that I responded very little to the rigor that Defender brought into the discussion. The reason why is that I considered it off topic, since I never claimed that partial differentiation cannot be made rigorous, just that its presentation is often unnecessarily confusing.