A Unipodal Experiment

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Abstract

I invented a problem to solve that I suspected the unipodal algebra could be used favorably to solve it. If you are not familair with the unipodal algebra, this website has a lot of material on it.

1 Problem

Statement of the problem:

Solve for the value of the expression (where x is real)

$$\left(x + \frac{1}{x}\right)^n - \left(x - \frac{1}{x}\right)^n \quad (n \text{ a positive integer})\,, \tag{1}$$

given the two relations

$$\left(x + \frac{1}{x}\right)^n + \left(x - \frac{1}{x}\right)^n = x\,, \tag{2a}$$

and

$$x^2 - \frac{1}{x^2} = B\,, \tag{2b}$$

where B is some nonzero real number.

Solution:

Let's define the unipode

$$a \equiv x + \frac{1}{x}u\,, \tag{3}$$

where we have used the standard basis $\{1, u\}$ $(u^2 = 1)$. Then, its unegate is

$$a^- = x - \frac{1}{x}u\,. \tag{4}$$

Taking the product of these last two equations, we get

$$aa^- = \big(x + \frac{1}{x}u\big)\big(x - \frac{1}{x}u\big) = x^2 - \frac{1}{x^2} = B\,, \tag{5}$$

where we used (2b).

Converting (3) to the idempotent basis, we get

$$a = \left(x + \frac{1}{x}\right) u_+ + \left(x - \frac{1}{x}\right) u_- \,, \tag{6}$$

On raising this last equation to the nth power, we get

$$a^n = \left(x + \frac{1}{x}\right)^n u_+ + \left(x - \frac{1}{x}\right)^n u_- \,. \tag{7}$$

Now, converting back to the standard basis, we have that

$$\begin{aligned} a^n &= \tfrac{1}{2}\Big[\left(x + \frac{1}{x}\right)^n + \left(x - \frac{1}{x}\right)^n\Big] + \tfrac{1}{2}\Big[\left(x + \frac{1}{x}\right)^n - \left(x - \frac{1}{x}\right)^n\Big]u \\ &= \tfrac{1}{2}x + \tfrac{1}{2}\Big[\left(x + \frac{1}{x}\right)^n - \left(x - \frac{1}{x}\right)^n\Big]u \,, \end{aligned} \tag{8}$$

where we used (2a). On taking the unegate of this last equation, we have that

$$(a^n)^- = (a^-)^n = \tfrac{1}{2}x - \tfrac{1}{2}\Big[\left(x + \frac{1}{x}\right)^n - \left(x - \frac{1}{x}\right)^n\Big]u \,. \tag{9}$$

On taking the product of (8) and (9), we have that

$$\begin{aligned} (a^n)(a^-)^n &= (aa^-)^n \\ &= \Big(\tfrac{1}{2}x + \tfrac{1}{2}\Big[\left(x + \frac{1}{x}\right)^n - \left(x - \frac{1}{x}\right)^n\Big]u\Big)\Big(\tfrac{1}{2}x - \tfrac{1}{2}\Big[\left(x + \frac{1}{x}\right)^n - \left(x - \frac{1}{x}\right)^n\Big]u\Big) \\ &= \tfrac{1}{4}x^2 - \tfrac{1}{4}\Big[\left(x + \frac{1}{x}\right)^n - \left(x - \frac{1}{x}\right)^n\Big]^2 = B^n \,, \end{aligned} \tag{10}$$

where we used (5).

Thus,

$$\left(x + \frac{1}{x}\right)^n - \left(x - \frac{1}{x}\right)^n = \sqrt{x^2 - 4B^n} \,. \tag{11}$$

Solving (2b) for x^2, we get

$$x^2 = \frac{B + \sqrt{B^2 + 4}}{2} \,. \tag{12}$$

So, finally, the answer is

$$\left(x + \frac{1}{x}\right)^n - \left(x - \frac{1}{x}\right)^n = \sqrt{\frac{B + \sqrt{B^2 + 4}}{2} - 4B^n} \,. \tag{13}$$