The Special Relativistic Addition of Velocities in the Unipodal Algebra

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Abstract

In this paper we present a proof of the Special Relativistic Addition of Velocities in the Unipodal Algebra for collinear relative motions. All motions are along the common x, x' axes. The approach here will use the *rapidity* parameter.

1 Introduction

The Special Relativistic Addition of Velocities formula looks like this

$$v'' = \frac{v + v'}{1 + v'v/c^2}\,, \tag{1}$$

where v is the speed of the S' frame relative to the S frame, and v' is the speed of some object in the S' frame. This formula contrasts greatly from that of the Galilean Relativistic Addition of Velocities formula, which looks like this

$$v'' = v + v'\,. \tag{2}$$

2 A Convenient Starting Point

I won't start from the beginning assumptions to develop the hyperbolic Lorentz transformation equation for SR relative motions along just one common axis, the x, x' axes. The Lorentz transformation equations take the form:

$$ct' = ct\cosh\theta - x\sinh\theta\,, \tag{3a}$$
$$x' = -ct\sinh\theta + x\cosh\theta\,, \tag{3b}$$

where c is the speed of light, t is the time in the S frame and x is the distance along the x-axis in the S frame, and similarly for t' and x' in the S' frame. Also, u is a formal unipodal element, not equal to any real number, such that $u^2 = 1$.

Now, we will capture the essence of these two transformation equations by the single unipodal equation

$$X' = X\, e^{-u\theta}\,, \tag{4}$$

where

$$X = x + ctu\,, \tag{5a}$$

$$X' = x' + ct'u\,, \tag{5b}$$

If the two origins coincide at $t = t' = 0$, then for some event after that in both frames, (x, ct) in S and (x', ct') in S', then

$$X'X'^{-} = XX^{-}\,, \tag{6}$$

where $u^{-} = -u$. From this we conclude that

$$x'^2 - c^2t'^2 = x^2 - c^2t^2\,. \tag{7}$$

I included this result to assure the reader that the unipodal version of the Lorentz transformation is legitimate.

By the way, it's easy enough to invert Eq. (4):

$$X = X'\, e^{u\theta}\,. \tag{8}$$

3 The Rapidity Parameter

To make use of the rapidity parameter θ for the Addition of Velocities formula, we first need to relate it to the velocity of the S' frame relative to the S frame.

We take the origin in S' as our coupling point. Its value in S' is $x' = 0$, but we have the relation of (3b), which gives us

$$x' = -ct\sinh\theta + x\cosh\theta\,, \tag{9}$$

which places a constraint on x, ct, provided by their ratio, through the equation

$$0 = -ct\sinh\theta + x\cosh\theta\,. \tag{10}$$

Since $v = x/t$, we get

$$\frac{v}{c} = \frac{x}{ct} = \frac{\sinh\theta}{\cosh\theta} = \tanh\theta\,. \tag{11}$$

4 Composition of Speeds

Let v' be the speed of some object as measured by the S'. We can imagine this object being the origin of a third reference frame S'', with transformation equation

$$X'' = X'\, e^{-u\theta'}\,. \tag{12}$$

By composing these three frames, we get

$$X'' = X\, e^{-u\theta}\, e^{-u\theta'} = X\, e^{-u(\theta+\theta')}\,, \tag{13}$$

where $\theta + \theta'$ parameterizes the speed of the object as seen from the S frame:

$$\frac{v''}{c} = \tanh\left(\theta + \theta'\right). \tag{14}$$

Gee, all we need now is some way to break up the hyperbolic tangent, such as

$$\tanh\left(\theta + \theta'\right) = \frac{\tanh\theta + \tanh\theta'}{1 + \tanh\theta\ \tanh\theta'}\,. \tag{15}$$

By substituting in and simplifying, we get

$$v'' = \frac{v + v'}{1 + v'v/c^2}\,, \tag{16}$$

which is what we were to prove.

By the way, it's not hard to show that, given that

$$\cosh^2\theta - \sinh^2\theta = 1\,, \tag{17}$$

then

$$\gamma = \cosh\,\theta = \frac{1}{\sqrt{1 - v^2/c^2}}\,. \tag{18}$$

5 Appendix

Some people define the rapidity as

$$y_v = \tfrac{1}{2}\ln\frac{1+v}{1-v}\,, \tag{19}$$

with $c = 1$. I want to show that this is equivalent to what we produced above.

Multiply (19) through by 2 and then exponentiate.

$$e^{2y_v} = \frac{1+v}{1-v}\,. \tag{20}$$

On solving this for v, we get

$$v = \frac{e^{2y_v} - 1}{e^{2y_v} + 1}\,. \tag{21}$$

Now multiply numerator and denominator by $e^{-2y_v}/2$ and we get

$$v = \frac{(e^{y_v} - e^{-y_v})/2}{(e^{y_v} + e^{-y_v})/2} = \tanh y_v\,, \tag{22}$$

where we used that

$$\sinh x = \frac{(e^x - e^{-x})}{2} \qquad \text{and} \qquad \cosh x = \frac{(e^x + e^{-x})}{2}\,. \tag{23}$$