A unipodal integral: $\int \cos x \cosh x \, dx$

by

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The purpose of this note is to prove the following identity

$$\int \cos x \cosh x \, dx = \tfrac{1}{2}(\cos x \sinh x + \sin x \cosh x)\,, \tag{1}$$

where x is real. Our approach here will be to embed the problem in the unipodal algebra, which is the extension of the complex numbers by the unipotent number u. That is, u is a noncomplex number that has square 1 and commutes with every complex number. For our purposes we will regard u as a unit vector.

The following list of identities is obeyed by the unipodal numbers

$$u^2 = 1\,, \tag{2a}$$
$$u_\pm \equiv \tfrac{1}{2}(1 \pm u)\,, \tag{2b}$$
$$u_+ + u_- = 1\,, \tag{2c}$$
$$u_\pm^2 = u_\pm\,, \tag{2d}$$
$$u u_\pm = \pm u_\pm\,, \tag{2e}$$
$$u = u_+ - u_-\,, \tag{2f}$$
$$u_+ u_- = 0\,, \tag{2g}$$
$$e^{xu} = \cosh x + u \sinh x\,, \tag{2h}$$
$$e^{-xu} = \cosh x - u \sinh x\,, \tag{2i}$$
$$\cosh x = \tfrac{1}{2}(e^{xu} + e^{-xu}) = \tfrac{1}{2}(e^{x} + e^{-x})\,, \tag{2j}$$
$$\sinh x = \tfrac{1}{2}u(e^{xu} - e^{-xu}) = \tfrac{1}{2}(e^{x} - e^{-x})\,, \tag{2k}$$
$$\cosh^2 x - \sinh^2 x = 1\,, \tag{2l}$$
$$u_\pm^- = u_\mp\,, \tag{2m}$$
$$u^{-1} = u\,. \tag{2n}$$

This next fundamental theorem can be proved with elementary power series expansion of the exponential

$$e^{z_+ u_+ + z_- u_-} = e^{z_+} u_+ + e^{z_-} u_-\,, \tag{3}$$

where z_+ and z_- are complex components of a unipode in idempotent basis. This last theorem is so fundamental that there are hardly proper words to indicate its true importance. And of similar importance is the relation

$$\mathrm{Log}\,(x) \equiv u_+ \mathrm{Log}\,(x_+) + u_- \mathrm{Log}\,(x_-)\,. \tag{4}$$

Every invertible unipodal number $a + xu$ can be written in the alternative form $\lambda e^{\nu u}$, where λ and ν are complex numbers. Thus

$$\lambda e^{\nu u} = a + xu\,. \tag{5}$$

DEFINITION. By the *unipodal conjugate* or *unegate* of the unipode $a_0 + a_1 u$ we mean the unipode $a_0 - a_1 u$. We define the following unegation operator to accomplish getting the unegate. Let x be any unipode then the unegate of x is x^-. Note that $xx^- = x^-x = a_0^2 - a_1^2$ is a complex number.

Now we can get to the proof of the given identity. Let x be real then

$$\int \cos x \cosh x \, dx = \text{Real} \int e^{ix} e^{ux} \, dx = \text{Real} \int e^{ix+ux} \, dx = \text{Real} \int e^{(i+u)x} \, dx \,. \tag{6}$$

Integrating this we get

$$\int \cos x \cosh x \, dx = \text{Real} \left\{ \frac{i-u}{-2} \, e^{(i+u)x} \right\} = \text{Real} \left\{ \tfrac{1}{2}(u-i) e^{ix} e^{ux} \right\} \,. \tag{7}$$

Now we expand the exponentials.

$$\int \cos x \cosh x \, dx = \text{Real} \left\{ \tfrac{1}{2}(u-i)(\cos x + i \sin x)(\cosh x + u \sinh x) \right\} . \tag{8}$$

On integrating and simplifying we get

$$\begin{aligned} &\int \cos x \cosh x \, dx \\ &\quad = \text{Real} \left\{ \tfrac{1}{2}(u-i)[\cos x \cosh x + \cos x \sinh x u + i \sin x \cosh x + iu \sin x \sinh x] \right\} \,. \end{aligned} \tag{9}$$

And finally we find the real parts to get

$$\int \cos x \cosh x \, dx = \tfrac{1}{2}(\cos x \sinh x + \sin x \cosh x) \,. \tag{10}$$