Inverse Hyperbolic Sine to Natural Logarithm

P. Reany

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A clue is anything that doesn't happen
the way it oughtta happen.
— Harry Orwell, TV
show *Harry O*

1 The Problem

Show that

$$\sinh^{-1} y = \ln \left[\, y + \sqrt{y^2+1}\,\right] . \tag{1}$$

2 The Prerequisites: The unipodal algebra

This algebra is formed as the extension of the complex numbers by the number u, where $u^2 = 1$, and u commutes with the complex numbers. The number u is said to be 'unipotent'. The set of numbers constructed this way are the unipodal numbers, a particular such number is called a unipode. The main conjugation operator on unipode a is the unegation operator, written a^-. It does not affect complex numbers, but it sends every u to its negative. Hence, if $a = x + yu$, where x, y are complex numbers, then $a^- = x - yu$. Unegation distributes over addition and multiplication.

The following are some properties that will come in handy:

$$u^2 = 1 \,, \tag{2a}$$
$$u_\pm \equiv \tfrac{1}{2}(1 \pm u) \,, \tag{2b}$$
$$u_\pm^2 = u_\pm \,, \tag{2c}$$
$$u = u_+ - u_- \,, \tag{2d}$$
$$u_+ u_- = 0 \,, \tag{2e}$$
$$u_+ + u_- = 1 \,, \tag{2f}$$
$$u u_+ = u_+ \,, \tag{2g}$$
$$u u_- = -u_- \,, \tag{2h}$$
$$(u_\pm)^- = u_\mp \,. \tag{2i}$$

You should prove (2c) – (2i). By the way, these two special unipodes $u_\pm$ square to themselves. Such numbers in a ring are referred to as *idempotents*. In the unipodal numbers they have no inverses. The fact that the unipodal number system is not a field is of little concern to me. In fact, most unipodes have inverses, so long as they are not multiples of one of the idempotents. If one needs field elements, the scalars of the unipodal numbers comprise the field of complex numbers.

Two often-used results are, for complex numbers w, z (which are used to convert unipodes between the bases $\{1, u\}$ and $\{u_+, u_-\}$):

$$w + zu = (w + z)u_+ + (w - z)u_- \,, \tag{3a}$$

$$wu_+ + zu_- = \tfrac{1}{2}(w + z) + \tfrac{1}{2}(w - z)u \,. \tag{3b}$$

Next, we learn how to take the 'norm' of a unipode. Let w be a unipode in standard basis, given by

$$w = a + bu \,, \tag{4}$$

where a, b are complex numbers. The 'norm' of w is given as[1]

$$ww^- = (a + bu)(a - bu) = a^2 - b^2 \,. \tag{5}$$

Now, let y be a unipode in idempotent basis, given as

$$y = Au_+ + Bu_- \,, \tag{6}$$

where A, B are complex numbers. The 'norm' of y is given as

$$\begin{aligned} yy^- &= (Au_+ + Bu_-)(Au_- + Bu_+) = ABu_+ + ABu_- \\ &= AB(u_+ + u_-) = AB \,. \end{aligned} \tag{7}$$

The unipodal algebra has two copies of the complex numbers, one for each component. In any true unipodal equation, the corresponding coefficients across the equal sign are equal to each other. This is similar to equating real and imaginary components across the equal sign in the complex algebra.

When I first used the unipodal algebra to solve polynomial equations (c. 1984-5), I used the Clifford 1 algebra over the complex numbers. The '1' means one unit vector u. So, a Clifford 1 number c can be represented as

$$c = a + bu \,, \tag{8}$$

where a, b are complex numbers. Of course, u being a unit vector, then

$$u^2 = 1 \,. \tag{9}$$

[1]Calling ww^- a 'norm' is rather imprecise. In accordance with terminology used by G. Sobczyk, I will call $\left| ww^- \right|^{1/2}$ the (unipodal) modulus, and ww^- the (unipodal) di-modulus of w. See the Appendix.

Now, the standard basis for this space is $\{1, u\}$ and the scalars are the complex numbers. To extract the 'scalar part' of (8), we use the selection operator $\langle \cdot \rangle$, as follows:

$$\langle\, c \,\rangle = \langle\, a + bu \,\rangle = a \,, \tag{10}$$

One can also subscript the selector with a zero for the scalar part:

$$\langle\, c \,\rangle_0 = \langle\, a + bu \,\rangle_0 = a \,, \tag{11}$$

and with a '1' for the vector part:

$$\langle\, c \,\rangle_1 = \langle\, a + bu \,\rangle_1 = bu \,. \tag{12}$$

When I adopted the name 'unipodal algebra' from a paper I cowrote with two other authors, I found a need to adopt new terminology for naming the scalar and vector parts. Just as complex numbers are composed of a real number times the unit '1' and another real number times the unit imaginary i, the unipodal numbers are composed of a complex number times the unit '1' and another complex number times the unipotent number u. The part of the unipode that does not contain the unipotent factor is called the 'complex part' of the unipode. The part that does contain the unipotent element factor is called the **uniplex part** of the unipode.

Now, before you complain that calling the scalar part of a unipode the 'complex part' is nonsense, I point out that in complex analysis, the nonimaginary part is referred to as the 'real part'. Lastly, when I say the 'uniplex part' in this series of papers, I refer only to the coefficient of the nonscalar part, which is complex only. Thus the uniplex part of unipode $c = a + bu$ is just b. Another way to think of the uniplex part of c is to take the scalar (or complex) part of cu.

$$\langle\, c \,\rangle_1 = \langle\, cu \,\rangle = \langle\, au + b \,\rangle = b \,. \tag{13}$$

Thus, one must be careful when I report I'm taking the uniplex part of a unipode (across all the papers I've written over the years), because at times it may contain that factor of u and at other times not. But like I said: In this series it will always mean only the scalar factor of the unipotent element.

3 The Solution

Logically, to prove this identity in the unipodal algebra, we should begin with an identity relation in the unipodal algebra that unites the exponential with hyperbolic trig functions, such as

$$e^{xu} = \cosh x + u \sinh x \,, \tag{14}$$

which can be proved similarly to how the Euler Identity is proved. So, to get started, let's pose a more general relationship in

$$\lambda e^{\theta u} = a + bu \,. \tag{15}$$

On multiplying this last equation by its unegate, we get

$$\lambda^2 = a^2 - b^2 \,, \tag{16}$$

which we'll need soon.

Now, taking the unipotent part of (15), we have that

$$\lambda \sinh\theta = b \,. \tag{17}$$

Solving for θ, we get

$$\theta = \sinh^{-1}(b/\lambda) \,. \tag{18}$$

Next, we multiply (15) through by u_+ and find that

$$\lambda e^\theta = a + b \,, \tag{19}$$

which we'll divide through by λ to get

$$e^\theta = \frac{a}{\lambda} + \frac{b}{\lambda} \,, \tag{20}$$

Our next big step is to set

$$y = \frac{b}{\lambda} \,. \tag{21}$$

Then (18) becomes

$$\theta = \sinh^{-1} y \,. \tag{22}$$

and (20) becomes

$$e^\theta = y + \frac{\sqrt{b^2 + \lambda^2}}{\lambda} = y + \sqrt{(b/\lambda)^2 + 1} = y + \sqrt{y^2 + 1} \,, \tag{23}$$

where we used (16). So, combining these last two equations, we have that

$$\sinh^{-1} y = \ln\left(y + \sqrt{y^2 + 1}\right) . \tag{24}$$