The Logarithm and Square Root of a Vector in the Unipodal Algebra

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Abstract

The Unipodal Algebra is used to take the Logarithm and square root of a vector. I don't know what these objects are (beyond their face-value appeal) or if they have a practical use, but if Euler's Identity is honored for being an interesting relation among important mathematical objects, both of these are as well. The $\operatorname{Log} u = i\pi u_-$.

1 Introduction

Euler's Identity is

$$e^{i\pi} + 1 = 0\,. \tag{1}$$

This is a duly magnificent relationship, no doubt. But perhaps the Log and square root of a vector is as interesting. The tool I'll wield for deriving these two oddities is the unipodal algebra.

The unipodal algebra is a commutative algebra that sits between the complex algebra on the low end and the quaternions and higher-dimensional Clifford algebras on the high end.

The unipodal algebra is formed by taking all linear combinations of the number 1 and the unipotent number u over the complex numbers, and then demand closure under multiplication.

A *unipotent number* is an entity that is neither $+1$ nor -1 that squares to 1. A unit vector, by definition, squares to 1.

Starting with unity and u we can define the two idempotent numbers:

$$u_+ = \tfrac{1}{2}(1+u) \qquad \text{and} \qquad u_- = \tfrac{1}{2}(1-u)\,. \tag{2}$$

Idempotents square to themselves. Try squaring them if you aren't convinced. Another interesting property of these idempotents are mutually annihilating. That is,

$$u_+u_- = 0\,. \tag{3}$$

One very useful property of these idempotents is that we can write any unipode $a + bu$ as a linear combination of them. In fact, we can write 1 as $u_+ + u_-$, and

u as $u_+ - u_-$ and thus convert a unipode number in the standard basis into a unipode in the idempotent basis.

$$a + bu = (a+b)u_+ + (a-b)u_- \,. \tag{4}$$

Our next task is to define the exponential function of unipode number. We define

$$\exp(a_+u_+ + b_-u_-) = e^{a_+}u_+ + e^{b_-}u_- \,, \tag{5}$$

where a_+ and b_- are complex numbers. We can also define the principal logarithm (Log) of a unipode as follows

$$\mathrm{Log}\,(a_+u_+ + b_-u_-) = u_+\,\mathrm{Log}\,a_+ + u_-\,\mathrm{Log}\,b_- \,, \tag{6}$$

where both a_+ and b_- are complex numbers, and the argument of Log is constrained to

$$-\pi < \arg \leq \pi \,. \tag{7}$$

We can now take the logarithm of a vector, say the unit vector u

$$\begin{aligned} \mathrm{Log}\,u &= \mathrm{Log}\,(1u_+ + (-1)u_-) \\ &= u_+\cancelto{0}{\mathrm{Log}\,1} + u_-\,\mathrm{Log}\,(-1) \\ &= u_-\,\mathrm{Log}\,(e^{i\pi}) \\ &= i\pi u_- \,, \end{aligned} \tag{8}$$

where Euler's Identity made a showing this time too.

Now for the square roots of u. First, we define nth roots of the unipodal number $a_+u_+ + b_-u_-$.

$$(a_+u_+ + b_-u_-)^{1/n} = (a_+)^{1/n}u_+ + (b_-)^{1/n}u_- \,. \tag{9}$$

Notice that since each nth root of a_+ and b_- can be taken independently of each other then there are n^2 nth roots represented by the Equation (9). Thus there will be four square roots of u.

$$u^{1/2} = (1u_+ + (-1)u_-)^{1/2} = (1)^{1/2}u_+ + (-1)^{1/2}u_- = (\pm 1)^{1/2}u_+ + (\pm i)^{1/2}u_- \,. \tag{10}$$

Two of the square roots are $u_+ + iu_-$ and its negative. And the other two are $u_+ - iu_-$ and its negative. So, what is a square root of a vector good for? I have no idea.

2 Conclusion

I first discovered the square root and logarithm of a vector in 1985 after playing around with hyperintegers, though I doubt that I was the first to discover them. I heard that Pertti Lounesto had also discovered them in the mid-1980s.