

# Main hyperbolic integral theorem

by

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The purpose of this note is to prove the following identity over the complex numbers

$$\int e^{xu} dx = ue^{xu}. \quad (1)$$

Our approach here will be to embed the problem in the unipodal algebra, which is the extension of the complex numbers by the unipotent number  $u$ . That is,  $u$  is a number not equal to any complex number but that has square 1 and commutes with every complex number. For our purpose we will regard  $u$  as a unit vector.

The following list of identities is obeyed by the unipodal numbers

$$u^2 = 1, \quad (2a)$$

$$u_{\pm} \equiv \frac{1}{2}(1 \pm u), \quad (2b)$$

$$u_+ + u_- = 1, \quad (2c)$$

$$u_{\pm}^2 = u_{\pm}, \quad (2d)$$

$$uu_{\pm} = \pm u_{\pm}, \quad (2e)$$

$$u = u_+ - u_-, \quad (2f)$$

$$u_+ u_- = 0, \quad (2g)$$

$$e^{xu} = \cosh x + u \sinh x, \quad (2h)$$

$$e^{-xu} = \cosh x - u \sinh x, \quad (2i)$$

$$\cosh x = \frac{1}{2}(e^{xu} + e^{-xu}) = \frac{1}{2}(e^x + e^{-x}), \quad (2j)$$

$$\sinh x = \frac{1}{2}u(e^{xu} - e^{-xu}) = \frac{1}{2}(e^x - e^{-x}), \quad (2k)$$

$$\cosh^2 x - \sinh^2 x = 1, \quad (2l)$$

$$u_{\pm}^{-} = u_{\mp}, \quad (2m)$$

$$u^{-1} = u. \quad (2n)$$

This next fundamental theorem can be proved with elementary power series expansion of the exponential

$$e^{z+u_+ + z-u_-} = e^{z+} u_+ + e^{z-} u_-. \quad (3)$$

This last theorem is so fundamental that there are hardly proper words to indicate its true importance. And of near similar importance is the relation

$$\text{Log}(x) \equiv u_+ \text{Log}(x_+) + u_- \text{Log}(x_-). \quad (4)$$

Every invertible unipodal number  $a + xu$  can be written in the alternative form  $\lambda e^{\nu u}$ . Thus

$$\lambda e^{\nu u} = a + xu \quad (5)$$

**DEFINITION.** By the *unipodal conjugate* or *unegate* of the  $a = a_0 + a_1 u$  we mean the unipode  $a_0 - a_1 u$ . We define the following unegation operator to accomplish getting the unegate. Let  $x$  be any unipode then the unegate of  $x$  is  $x^-$ . Note that  $xx^- = x^-x = a_0^2 - a_1^2$  is a complex number.

Now we can get to the proof of the given identity. Let

$$\int e^{xu} dx = \int e^{xu_+ - xu_-} dx = \int e^x u_+ dx + \int e^{-x} u_- dx = u_+ \int e^x dx + u_- \int e^{-x} dx. \quad (6)$$

We have reduced the integral of a unipodal number to the integrals of familiar complex numbers. Our next step gives us

$$\int e^{xu} dx = e^x u_+ - e^{-x} u_- = \frac{1}{2}(1+u)e^x - \frac{1}{2}(1-u)e^{-x}. \quad (7)$$

Collecting complex and complex vector parts gives

$$\int e^{xu} dx = \frac{1}{2}(e^x - e^{-x}) + u \frac{1}{2}(e^x + e^{-x}) = \sinh x + u \cosh x. \quad (8)$$

Now we just virtually emplace a factor of  $u$  to get

$$\int e^{xu} dx = u(\cosh x + u \sinh x) = ue^{ux}. \quad (9)$$