Normal Modes of a Loaded String by the Unipodal Algebra

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Abstract

Our job is to find the normal mode solutions to two particles of the same mass connected by springs and sliding frictionlessly on a level surface.

1 Introduction

The problem of finding the normal modes of vibrations of a loaded string of mass particles, each interacting with its neighbor, is of great interest to physicists and engineers. In general, the string can be loaded with n particles, but in our case here, we consider only two particles, as in the figure below.

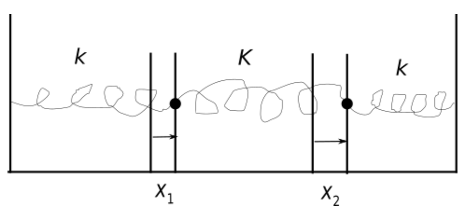

Figure 1. We are ignoring the effects of gravity here. The only forces on the two mass particles, each of mass m, is due to the massless springs attached. Values x_1 and x_2 represent the displacements of the particles from their equilibrium positions. Values k and K represent spring constants.

Our goal is to set down the equations of motion of the particles, place them in matrix form, and then solve for the normal vibrational modes. However, we will not solve this matrix equation. If you prefer to see that solution, you need only consult may references on the Internet or in some appropriate physics or engineering text. In this paper, we will see the novel solution by use of the unipodal algebra, or perhaps I should say the unipodal calculus.

Definition: A vibration of a collection of particles is said to be in **normal mode** when all particles move sinusoidally and with the same frequency and with a fixed phase relation (Wikipedia).

Now we calculate the forces on each mass point due to the attached springs.

$$F_1 = -kx_1 - K[-(x_2 - x_1)]\,,$$
$$F_2 = -kx_2 - K(x_2 - x_1)\,.$$

On reorganizing and setting $F_i = m\ddot{x}_i$, we have that

$$m\ddot{x}_1 = -(k+K)x_1 + Kx_2\,,$$
$$m\ddot{x}_2 = \quad Kx_1 - (k+K)x_2\,,$$

which can be written in matrix form as

$$m\frac{d^2}{dt^2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -(k+K) & K \\ K & -(k+K) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\,. \qquad (1)$$

2 Getting to know the Unipodal Algebra

The unipodal algebra is formed by all linear combinations of the two basis vectors $\{1, u\}$ (forming the *standard basis*), where u is some number not ± 1 whose square is unity, making it a unipotent number.

$$u^2 = 1\,. \qquad (2)$$

There is an alternative basis, called the *idempotent basis*, which is given by $\{u_+, u_-\}$, which will be defined below.

Now, for some easy identities. The better you know these identities, the better you'll be able to follow the proof.

$$u_\pm = \tfrac{1}{2}(1 \pm u), \qquad (3)$$
$$u_+ + u_- = 1, \qquad (4)$$
$$u_\pm^2 = u_\pm, \qquad (5)$$
$$uu_\pm = \pm u_\pm, \qquad (6)$$
$$u = u_+ - u_-, \qquad (7)$$
$$u_+u_- = 0\,, \qquad (8)$$
$$(a_0 + a_1 u)u_+ = (a_0 + a_1)u_+\,, \qquad (9)$$
$$(a_0 + a_1 u)u_- = (a_0 - a_1)u_-\,. \qquad (10)$$

The following theorem can be proved with elementary power series expansion of the exponential

$$e^{z_+u_+ + z_-u_-} = e^{z_+}u_+ + e^{z_-}u_-. \qquad (11)$$

3 Solving the Generic Unipodal Equation

Before we attemt to solve Eq. (1), let's first solve the problem with generic complex coefficients.

The pair of equations

$$m\ddot{x}_1 = a_0 x_1 + a_1 x_2\,,$$
$$m\ddot{x}_2 = a_1 x_1 + a_0 x_2\,,$$

can be written in matrix form as

$$m\frac{d^2}{dt^2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_0 & a_1 \\ a_1 & a_0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\,. \tag{12}$$

With a little algebra, one can show that the same information as contained in (12) is contained in its unipodal equivalent

$$m\frac{d^2}{dt^2}X = AX\,, \tag{13}$$

where

$$X = x_1 + x_2 u \qquad \text{and} \qquad A = a_0 + a_1 u. \tag{14}$$

So, let's go to cases:

Case 1) $x_1 = x_2 = \bar{x}$. (the symmetric solution)

Case 2) $-x_2 = x_1 = \bar{x}$. (the antisymmetric solution)

Case 3) $|\,x_1\,| \neq |\,x_2\,|$.

In Cases 1 and 2, we find that the problems quickly reduce to a one variable problem, in the former case because X reduces to a multiple of u_+, and in the latter case, because X reduces to a multiple of u_-. In both of these case X has no inverse, but that's not a problem. The condition provided in Case 3 guarantees that X does have an inverse. We won't need to calculate the inverse, but we'll use its existence to further the solution.

Case 1) $x_1 = x_2 = \bar{x}$. Substituting these into (13), we get

$$m\frac{d^2}{dt^2}\bar{x}(1+u) = A\bar{x}(1+u) = A_+\bar{x}(1+u)\,, \tag{15}$$

where $A_+ = (a_0 + a_1)$.[1] On simplifying, we get

$$m\frac{d^2\bar{x}}{dt^2} = A_+\bar{x}\,, \tag{16}$$

[1] Technically, we should divide through by 2 to replace $1+u$ by u_+.

which is a one-variable problem, with ansatz

$$\bar{x} = ce^{\omega t}\,, \tag{17}$$

where c and ω are a complex numbers. On substituting this into (16), we get

$$mc\omega^2 e^{\omega t} = A_+ ce^{\omega t}\,. \tag{18}$$

After some cancellation:

$$m\omega^2 = A_+\,. \tag{19}$$

Therefore,

$$\omega_\pm = \pm\sqrt{\frac{A_+}{m}}\,. \tag{20}$$

so,

$$\bar{x} = c_1 e^{\sqrt{\frac{A_+}{m}}\,t} + c_2 e^{-\sqrt{\frac{A_+}{m}}\,t}\,, \tag{21}$$

where, again, $A_+ = a_0 + a_1$, and c_1 and c_2 are arbitrary complex numbers.

Case 2) $-x_2 = x_1 = \bar{x}$. Substituting these into (13), we get

$$\bar{x} = c_1 e^{\sqrt{\frac{A_-}{m}}\,t} + c_2 e^{-\sqrt{\frac{A_-}{m}}\,t}\,, \tag{22}$$

where $A_- = a_0 - a_1$. The omitted steps are very similar to the steps of the previous case, except that the quantity $(1+u)$ in (15) was replaced by $(1-u)$, accounting for why this time we get A_- rather than A_+.

Case 3) $|\,x_1\,| \neq |\,x_2\,|$. In this case, X will never be a scalar multiple of an idempotent, hence it will always have an inverse. We will apply that inverse after we have applied the derivatives to the ansatz

$$X = ce^{\omega t}\,, \tag{23}$$

where c and ω are a unipodal numbers:

$$\omega = \omega_0 + \omega_1 u = \omega_+ u_+ + \omega_- u_-\,. \tag{24}$$

Performing the ansatz substitution, we get

$$m\omega^2 X = AX\,, \tag{25}$$

which reduces to

$$m\omega^2 = A\,. \tag{26}$$

There are many ways to proceed at this point to solve for ω_0 and ω_1. We could solve for ω by solving (26) for ω^2 and then taking the square root of A/m, but I prefer not to proceed this way because taking square roots of unipodal numbers can complicate matters by introducing extraneous roots. Presented for your consideration is the following.

On multiplying through by first u_+ and then by u_-, we get, respectively,

$$m\omega_+^2 = A_+ \,, \tag{27a}$$
$$m\omega_-^2 = A_- \,. \tag{27b}$$

Solving these two for ω_+ and ω_-, we get

$$\omega_+ = \pm\sqrt{A_+/m}\,, \tag{28a}$$
$$\omega_- = \pm\sqrt{A_-/m}\,. \tag{28b}$$

By expanding (23) into its idempotent basis, we can avoid solving for ω_0 and ω_1. So, we get

$$x_+u_+ + x_-u_- = (c_+u_+ + c_-u_-)(e^{\omega_+ t}u_+ + e^{\omega_-\,t}u_-)\,. \tag{29}$$

From which we get that

$$x_+ = c_+\,e^{\omega_+\,t}\,, \tag{30a}$$
$$x_- = c_-\,e^{\omega_-\,t}\,. \tag{30b}$$

Finally, we get the displacements

$$x_1 = c_1e^{\omega_+\,t} + c_2e^{\omega_-\,t}\,, \tag{31a}$$
$$x_2 = c_1e^{\omega_+\,t} - c_2e^{\omega_-\,t}\,, \tag{31b}$$

where we made the substitutions: $c_1 = c_+/2$ and $c_2 = c_-/2$.

4 Solving the Specific Unipodal Equation

Now we return **Case 1)** with $x_1 = x_2$, which means that both particles have the exact same motion, but is it normal mode? We have that

$$a_0 = -(k+K) \qquad \text{and} \qquad a_1 = K\,. \tag{32}$$

Therefore,

$$A_+ = a_0 + a_1 = -k\,. \tag{33}$$

On substituting A_+ value into (21), we get

$$\bar{x} = c_1'e^{i\sqrt{\frac{k}{m}}\,t} + c_2'e^{-i\sqrt{\frac{k}{m}}\,t}\,. \tag{34}$$

We can find a sinusoidal solution by setting the constants equal, such as $c_1' = c_2' = c/2$, then

$$\bar{x} = \frac{c}{2}\Big[e^{i\sqrt{\frac{k}{m}}\,t} + e^{-i\sqrt{\frac{k}{m}}\,t}\Big] = c\cos\big(\sqrt{\tfrac{k}{m}}\,t\big)\,. \tag{35}$$

Thus, both x_1 and x_2 are sinusoidal and have the same frequency.

More generally, we can let $c'_1 = \frac{1}{2}ce^{i\phi}$ and $c'_2 = \frac{1}{2}ce^{-i\phi}$

$$\bar{x} = \frac{c}{2}\Big[e^{i\left(\sqrt{\frac{k}{m}}\,t+\phi\right)} + e^{-i\left(\sqrt{\frac{k}{m}}\,t+\phi\right)}\Big] = c\cos\big(\sqrt{\tfrac{k}{m}}\,t+\phi\big)\,. \tag{36}$$

Now we return **Case 2)** with $x_1 = -x_2$, which means that both particles have a mirror-image motion, but is it a normal mode? We have that

$$a_0 = -(k+K) \qquad \text{and} \qquad a_1 = K\,, \tag{37}$$

Therefore,

$$A_- = a_0 - a_1 = -(k+2K)\,. \tag{38}$$

On substituting A_- value into (22), we get

$$\bar{x} = c_1 e^{i\sqrt{\frac{k+2K}{m}}\,t} + c_2 e^{-i\sqrt{\frac{k+2K}{m}}\,t}\,. \tag{39}$$

On following the same reasoning as in Case 1, then

$$\bar{x} = \frac{c}{2}\Big[e^{i\left(\sqrt{\frac{k+2K}{m}}\,t+\phi\right)} + e^{-\left(\sqrt{\frac{k+2K}{m}}\,t+\phi\right)}\Big] = c\cos\big(\sqrt{\tfrac{k+2K}{m}}\,t+\phi\big)\,. \tag{40}$$

Thus, both x_1 and x_2 are sinusoidal and have the same frequency, but as the mirror images of each other.

Now we return to **Case 3)**.

With the ansatz

$$X = ce^{\omega t}\,, \tag{41}$$

we calculated the displacements

$$x_1 = c_1 e^{\omega_+\,t} + c_2 e^{\omega_-\,t}\,, \tag{42a}$$

$$x_2 = c_1 e^{\omega_+\,t} - c_2 e^{\omega_-\,t}\,. \tag{42b}$$

We know from above that

$$\omega_+ = \pm i\sqrt{k/m}\,, \tag{43a}$$

$$\omega_- = \pm i\sqrt{(k+2K)/m}\,. \tag{43b}$$

Therefore,

$$x_1 = c_1 e^{\pm i\sqrt{k/m}\,t} + c_2 e^{\pm i\sqrt{(k+2K)/m}\,t}\,, \tag{44a}$$

$$x_2 = c_1 e^{\pm i\sqrt{k/m}\,t} - c_2 e^{\pm i\sqrt{(k+2K)/m}\,t}\,. \tag{44b}$$

Of course, x_1 and x_2 are displacements, so they are real numbers. Here, we'll restrict c_1 and c_2 to be real. This forces us to set the imaginary parts to zero:

$$\text{case } x_1\text{:} \quad (\pm)c_1 \sin\left(\sqrt{k/m}\, t\right) + (\pm)c_2 \sin\left(\sqrt{(k+2K)/m}\, t\right) = 0\,, \tag{45a}$$

$$\text{case } x_2\text{:} \quad (\pm)c_1 \sin\left(\sqrt{k/m}\, t\right) - (\pm)c_2 \sin\left(\sqrt{(k+2K)/m}\, t\right) = 0\,. \tag{45b}$$

In either case, by sign choices, it comes down to establishing particles motions consistent with

$$c_1 \sin\left(\sqrt{k/m}\, t\right) = \pm c_2 \sin\left(\sqrt{(k+2K)/m}\, t\right), \tag{46}$$

which cannot be done for all t. Consequently, Case 3 will not give us additional normal modes of vibration.

5 Afterthoughts

In general, one would approach a problem as we solved above with eigenvalues and eigenvectors, which can be used for $n \times n$ matrices, rather than the special 2×2 matrix we developed. In fact, the matrix

$$M = \begin{bmatrix} -(k+K) & K \\ K & -(k+K) \end{bmatrix} \tag{47}$$

is rather special even among 2×2 matrices, for it's of the form

$$M = \alpha \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \beta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{48}$$

where $\alpha = -(k+K)$ and $\beta = K$. We can form a matrix form of the unipodal algebra by taking all linear complex multiples of

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{49}$$

where I corresponds to the number 1 and J corresponds to the number u, because $J^2 = 1$. (Try it, if you doubt it.)

The eigenvalues of M are

$$\lambda_1 = -k \qquad \text{and} \qquad \lambda_2 = -(k+2K)\,, \tag{50}$$

and has corresponding eigenvectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \tag{51}$$

If we take $M - \lambda_1 I$, we get

$$M - \lambda_1 I = \begin{bmatrix} -K & K \\ K & -K \end{bmatrix} = K \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = K(I - J)\,. \tag{52}$$

To get the eigenvector v_1, we're supposed to solve this equation

$$(M - \lambda_1 I)v_1 = K(I - J)v_1 = 0\,. \tag{53}$$

Now, we're not looking for a unique v_1, in the sense that if we found any v_1 that satisfied (53), then any nonzero scalar multiple of it would also satisfy it. For a similar reason, we don't need to know what K is, so long as it's not zero, so we rewrite (53) as

$$(I - J)v_1 = 0\,. \tag{54}$$

Now, if v_1 were a 2×2 matrix, we could satisfy this last equation by setting

$$v_1 = (I + J)\,, \tag{55}$$

since $(I - J)$ and $(I + J)$ are mutually annihilating idempotents.

If we think of the unipodal numbers as a 2-dimensional vector space over the complex numbers, with 1 and u as basis elements of this vector spaces, we can represent it in matrix form by the correspondence, for arbitrary unipodal number $p + qu$,

$$p + qu \Longleftrightarrow \begin{bmatrix} p \\ q \end{bmatrix}\,, \tag{56}$$

Applying this correspondence, we have

$$u_+ = \tfrac{1}{2}(1 + u) \Longleftrightarrow \tfrac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad u_- = \tfrac{1}{2}(1 - u) \Longleftrightarrow \tfrac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\,. \tag{57}$$

Now, we have the interpretation that the 2×1 vector that annihilates $I - J$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the 2×1 vector that annihilates $I + J$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. So, we have this strange admixture of representations in which mutually annihilating idempotents taken one from 2×2 matrices and another from 2×1 matrices. Perhaps the mystery is resolved a bit by noticing that each row of the idempotent matrix $I - J$ or $I + J$ is annihilating to the eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, respectively.

We note that the unipodal algebra in matrix form is a subalgebra of the Pauli Algebra, where J would be labeled as either σ_1 or σ_x.

6 Conclusion

We have solved for the known normal mode vibrations of a dual loaded string without finding eigenvalues and eigenvectors, specifically (such as invoking determinants and the like). We also proved that there are no other normal modes than given by the two idempotent solutions above, and such a proof seems lacking in the common literature.

One always hopes that one can generalize methods, and we shall see. If I had not kept the faith in the unipodal algebra to produce ever more interesting mathematics, I would have never discovered unipodal integral solutions to integration problems or the interesting unipodal version of the Chebyshev polynomials.

7 Appendix: Naive Application of the Laplace Transform

Above, we've solved the following equation

$$m\frac{d^2}{dt^2}X = AX\,, \tag{58}$$

with an ansatz. Now, we'll naively apply the Laplace transformation, as though there are no complications to deal with. Basically, we'll treat (58) as if it were an equation in one real variable, and see where it takes us. (This is my first attempt at applying the Laplace transform to a unipodal equation.)

First, we'll add in some convenient initial conditions:

$$X(0) = 0 \qquad \text{and} \qquad X'(0) = V_0\,. \tag{59}$$

So, then taking the Laplace transform of (58), we get

$$m[s^2X(s) - V_0] = AX(s)\,, \tag{60}$$

On solving for $X(s)$, we get

$$X(s) = \frac{V_0}{s^2 - A/m} = \frac{V_0}{2}\sqrt{\frac{m}{A}}\Big[\frac{1}{s - \sqrt{A/m}} - \frac{1}{s + \sqrt{A/m}}\Big]\,. \tag{61}$$

On taking the inverse Laplace transformation, we get

$$X(t) = \frac{V_0}{2}\sqrt{\frac{m}{A}}\big[e^{\sqrt{A/m}\,t} - e^{-\sqrt{A/m}\,t}\big]\,. \tag{62}$$

Everything is simpler in the idempotent basis. First,

$$\begin{aligned} e^{\sqrt{A/m}\,t} - e^{-\sqrt{A/m}\,t} &= (e^{\sqrt{A_+/m}\,t} - e^{-\sqrt{A_+/m}\,t})u_+ \\ &\quad + (e^{\sqrt{A_-/m}\,t} - e^{-\sqrt{A_-/m}\,t})u_- \qquad (63)\\ &= 2\sinh(\sqrt{A_+/m}\,t)\,u_+ + 2\sinh(\sqrt{A_-/m}\,t)\,u_-\,. \qquad (64)\end{aligned}$$

Second,

$$\begin{aligned}\sqrt{\frac{m}{A}} &= \sqrt{(A/m)^{-1}} \\ &= \sqrt{[(A_+/m)u_+ + (A_-/m)u_-]^{-1}} \\ &= \sqrt{[(m/A_+)u_+ + (m/A_-)u_-]} \\ &= \pm\sqrt{m/A_+}u_+ + (\pm)\sqrt{m/A_-}u_- \end{aligned} \tag{65}$$

Hence, (62) becomes

$$\begin{aligned} X_+(t)u_+ + X_-(t)u_- &= (V_{0+}u_+ + V_{0-}u_-)[\pm\sqrt{m/A_+}u_+ + (\pm)\sqrt{m/A_-}u_-] \\ &\quad \times\big[\sinh(\sqrt{A_+/m}\,t)\,u_+ + \sinh(\sqrt{A_-/m}\,t)\,u_-\big]\,. \end{aligned} \tag{66}$$

From this we have that

$$X_+(t) = x_1 + x_2 = \pm V_{0+}\sqrt{m/A_+}\,\sinh(\sqrt{A_+/m}\,t)\,, \tag{67a}$$

$$X_-(t) = x_1 - x_2 = (\pm)V_{0-}\sqrt{m/A_-}\,\sinh(\sqrt{A_-/m}\,t)\,. \tag{67b}$$

Therefore,

$$x_1 = \pm\frac{V_{0+}}{2}\sqrt{m/A_+}\,\sinh\left(\sqrt{A_+/m}\,t\right) + (\pm)\frac{V_{0-}}{2}\sqrt{m/A_-}\,\sinh\left(\sqrt{A_-/m}\,t\right), \tag{68a}$$

$$x_2 = \pm\frac{V_{0+}}{2}\sqrt{m/A_+}\,\sinh\left(\sqrt{A_+/m}\,t\right) - (\pm)\frac{V_{0-}}{2}\sqrt{m/A_-}\,\sinh\left(\sqrt{A_-/m}\,t\right). \tag{68b}$$

With

$$A_+ = -k \qquad \text{and} \qquad A_- = -(k+2K)\,, \tag{69}$$

we get

$$x_1 = \pm\frac{1}{2}\Big[V_{0+}\sqrt{m/k}\,\sin(\sqrt{\frac{k}{m}}\,t) + (\pm)V_{0-}\sqrt{\frac{m}{k+2K}}\,\sin\big(\sqrt{\frac{k+2K}{m}}\,t\big)\Big]\,, \tag{70a}$$

$$x_2 = \pm\frac{1}{2}\Big[V_{0+}\sqrt{m/k}\,\sin(\sqrt{\frac{k}{m}}\,t) - (\pm)V_{0-}\sqrt{\frac{m}{k+2K}}\,\sin\big(\sqrt{\frac{k+2K}{m}}\,t\big)\Big]\,. \tag{70b}$$

Assuming that I haven't made any algebraic errors so far, we can try to find solutions from these for our special symmetric and antisymmetric cases. In the former case, $x_1 = x_2$, hence

$$x_1 = x_2 = \pm\frac{V_{0+}}{2}\sqrt{m/k}\,\sin\big(\sqrt{\frac{k}{m}}\,t\big)\,. \tag{71a}$$

And, in the antisymmetric case, $x_1 = -x_2$, hence

$$x_1 = -x_2 = (\pm)\frac{V_{0-}}{2}\sqrt{\frac{m}{k+2K}}\,\sin\big(\sqrt{\frac{k+2K}{m}}\,t\big)\,. \tag{71b}$$

I suppose the ambiguity of sign in Eqs. (71a) and (71b) can be dealt with by resolving what are the extraneous roots.

Alternative Approach

However, if we multiply (58) through by u_+, we get

$$m\frac{d^2}{dt^2}X_+ = A_+X_+\,, \tag{72}$$

with $X_+(0) = X_+(0) = 0$, $X'_+(0) = V_{0+}$, and $X'_-(0) = V_{0-}$. On taking the Laplace transformation of (72), we get

$$X_+(t) = \frac{V_{0+}}{2}\sqrt{\frac{m}{A_+}}\big[e^{\sqrt{A_+/m}\,t} - e^{-\sqrt{A_+/m}\,t}\big]\,. \tag{73}$$

And, if we multiply (58) through by u_-, we get

$$m\frac{d^2}{dt^2}X_- = A_- X_- \,, \tag{74}$$

with similar solution

$$X_-(t) = \frac{V_{0-}}{2}\sqrt{\frac{m}{A_-}}\left[e^{\sqrt{A_-/m}\,t} - e^{-\sqrt{A_-/m}\,t}\right]. \tag{75}$$

From these we get

$$x_1 + x_2 = -V_{0+}\sqrt{\frac{m}{k}}\sin\sqrt{k/m}\,t\,, \tag{76a}$$

$$x_1 - x_2 = -V_{0-}\sqrt{\frac{m}{k+2K}}\sin\sqrt{(k+2K/m)}\,t\,. \tag{76b}$$

Now, we return to our special cases. First, the symmetric case: $x_1 = x_2$. In this case $V_{0-} = 0$. So, from (76a) we have that

$$x_1 = x_2 = -\tfrac{1}{2}V_{0+}\sqrt{\frac{m}{k}}\sin\sqrt{k/m}\,t\,. \tag{77a}$$

Second, in the antisymmetric case: $x_1 = -x_2$, $V_{0+} = 0$. So, from (76b) we have that

$$x_1 = -x_2 = -\tfrac{1}{2}V_{0-}\sqrt{\frac{m}{k+2K}}\sin\sqrt{(k+2K/m)}\,t\,. \tag{77b}$$

Our latter solutions in (77a) and (77b) compare favorably with those of (71a) and (71b)