

Quadratic Unipodal

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The quadratic equation looks like this

$$ax^2 + bx + c = 0. \tag{1}$$

If a is different from zero, we can divide this equation through by a to get

$$x^2 + b'x + c' = 0, \tag{2}$$

where $b' = b/a$ and $c' = c/a$. A polynomial whose coefficient of its highest order term is unity (or rather 1) is called *monic*. Clearly Equations (1) and (2) are equivalent and have the same two roots. So solving for the roots to the general quadratic equation in (1) is equivalent to solving for the roots to (2). The solution for (2) is

$$x = -b'/2 \pm \sqrt{b'^2/4 - c'}, \tag{3}$$

which takes the form

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{4}$$

as the solution to (1).

Our purpose in this paper is to show an alternative method to solve for the roots of the quadratic than to use the method of “completing the square” as it is commonly done. I will demonstrate the use of the unipodal algebra to solve for the roots of the quadratic in a manner similar to that which I first did by solving for the roots to the cubic equation in 1985.

There is a certain beauty and simplicity of the solution to the proof of the unipodal quadratic formula, but it comes with a cost. Just the same, the unipodal solution I’ll present here goes like this: Given the unipodal quadratic equation

$$X^2 = C, \tag{5}$$

if we just expand it down to components, we’ll get a quadratic equation in complex variables; but if we take its square root, we’ll get the roots to the same quadratic equation.

First, however, a little explanation of the unipodal algebra might help. Every unipodal number is the linear combination of the two basis “vectors” $\{1, u\}$ over the complex numbers. The number u is different from any complex number

and it has the two properties: 1) $u^2 = 1$ and 2) u commutes with every complex number. This makes the unipodal algebra a commutative extension of the complex algebra. Now, every unipodal number can be written in the *standard form* $X = a + bu$ where a and b are complex numbers. But we can define the two idempotent numbers $u_{\pm} \equiv \frac{1}{2}(1 \pm u)$. An *idempotent number* is a number that squares to itself. Also, it is easy to prove that $u = u_+ - u_-$. The reader should be able to prove that $1 = u_+ + u_-$. With these definitions and relations we can convert any number in standard form into a number in “idempotent form” $Au_+ + Bu_-$. Further, we can go from idempotent form to standard form too.

So, our present solution begins with the unipodal equation $X^2 = C$, where both X and C are unipodal numbers. Specifically,

$$X = x_0 + x_1u \quad \text{and} \quad C = c_0 + c_1u. \quad (6)$$

From this we can take the square root of both sides of $X^2 = C$ to get the four roots

$$X = (\pm) \begin{pmatrix} 1 \\ u \end{pmatrix} [\sqrt{c_0 + c_1}u_+ + \sqrt{c_0 - c_1}u_-], \quad (7)$$

from which we get

$$x_0 + x_1u = (\pm) \begin{cases} \sqrt{c_0 + c_1}u_+ + \sqrt{c_0 - c_1}u_- , \\ \sqrt{c_0 + c_1}u_+ - \sqrt{c_0 - c_1}u_- . \end{cases} \quad (8)$$

On flipping the basis, we have that

$$x_0 + x_1u = (\pm) \begin{cases} \frac{1}{2}[\sqrt{c_0 + c_1} + \sqrt{c_0 - c_1}] + u \frac{1}{2}[\sqrt{c_0 + c_1} - \sqrt{c_0 - c_1}], \\ \frac{1}{2}[\sqrt{c_0 + c_1} - \sqrt{c_0 - c_1}] + u \frac{1}{2}[\sqrt{c_0 + c_1} + \sqrt{c_0 - c_1}]. \end{cases} \quad (9)$$

Thus both x_0 and x_1 have four roots each. Our solution is really quite simple. We will use only x_0 for our quadratic and reduce the number of roots in half by using x_0^2 instead of x_0 . Doing this we get that

$$(x_0^2)_{\pm} = \frac{1}{2}(c_0 \pm \sqrt{c_0^2 - c_1^2}). \quad (10)$$

OK, we have a form that already looks very close to the standard quadratic formula! Now all we need is the quadratic equation that x_0^2 belongs to.

We’ll get this equation from expanding $x^2 = c$ and equating complex and uniplex parts¹ to get

$$\begin{cases} x_0^2 + x_1^2 = c_0 , \\ 2x_0x_1 = c_1 . \end{cases} \quad (11)$$

Eliminating x_1 between these two equations yields

$$(x_0^2)^2 - c_0x_0^2 + c_1^2/4 = 0. \quad (12)$$

¹Every unipodal number can be written in the form $a + bu$ where a is a complex number referred to as the “complex part” of the unipodal number, and where b is a complex number referred to as the “uniplex part” of the unipodal number.

Now we compare our quadratic (12) to a generic monic quadratic

$$y^2 + \alpha y + \beta = 0, \quad (13)$$

where α and β are any complex numbers. Comparing coefficients we get

$$\begin{cases} c_0 = -\alpha, \\ c_1^2 = 4\beta. \end{cases} \quad (14)$$

Thus we arrive at the quadratic formula by substituting these values into (10) to get

$$y_{\pm} = \frac{1}{2}(-\alpha \pm \sqrt{\alpha^2 - 4\beta}). \quad (15)$$

Substituting $\alpha = b/a$ and $\beta = c/a$ we get the same form as in (4).