

Olympiad Problem 1

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Abstract

I invented a problem to solve that I suspected the unipodal algebra could be used favorably to solve it. If you are not familiar with the unipodal algebra, this website has a lot of material on it.

You cannot ask us to take sides against arithmetic.

— Winston Churchill

1 Problem

Statement of the problem:

Solve for the value of the expression (where x is real)

$$\left(x + \frac{1}{x}\right)^n - \left(x - \frac{1}{x}\right)^n \quad (n \text{ a positive integer}), \quad (1)$$

given the two relations

$$\left(x + \frac{1}{x}\right)^n + \left(x - \frac{1}{x}\right)^n = x, \quad (2a)$$

and

$$x^2 - \frac{1}{x^2} = B, \quad (2b)$$

where B is some nonzero real number.

Solution:

Let's define the unipode

$$a \equiv x + \frac{1}{x}u, \quad (3)$$

where we have used the standard basis $\{1, u\}$ ($u^2 = 1$). Then, its unegate is

$$a^- = x - \frac{1}{x}u. \quad (4)$$

Taking the product of these last two equations, we get

$$aa^- = \left(x + \frac{1}{x}u\right)\left(x - \frac{1}{x}u\right) = x^2 - \frac{1}{x^2} = B, \quad (5)$$

where we used (2b). [I call aa^- the *di-modulus* of a .]

Converting (3) to the idempotent basis, we get

$$a = \left(x + \frac{1}{x}\right)u_+ + \left(x - \frac{1}{x}\right)u_-, \quad (6)$$

On raising this last equation to the n th power, we get

$$a^n = \left(x + \frac{1}{x}\right)^n u_+ + \left(x - \frac{1}{x}\right)^n u_-. \quad (7)$$

Now, converting back to the standard basis, we have that

$$\begin{aligned} a^n &= \frac{1}{2} \left[\left(x + \frac{1}{x}\right)^n + \left(x - \frac{1}{x}\right)^n \right] + \frac{1}{2} \left[\left(x + \frac{1}{x}\right)^n - \left(x - \frac{1}{x}\right)^n \right] u \\ &= \frac{1}{2}x + \frac{1}{2} \left[\left(x + \frac{1}{x}\right)^n - \left(x - \frac{1}{x}\right)^n \right] u, \end{aligned} \quad (8)$$

where we used (2a). On taking the unegate of this last equation, we have that

$$(a^n)^- = (a^-)^n = \frac{1}{2}x - \frac{1}{2} \left[\left(x + \frac{1}{x}\right)^n - \left(x - \frac{1}{x}\right)^n \right] u. \quad (9)$$

On taking the product of (8) and (9), we have that

$$\begin{aligned} (a^n)(a^-)^n &= (aa^-)^n \\ &= \left(\frac{1}{2}x + \frac{1}{2} \left[\left(x + \frac{1}{x}\right)^n - \left(x - \frac{1}{x}\right)^n \right] u\right) \left(\frac{1}{2}x - \frac{1}{2} \left[\left(x + \frac{1}{x}\right)^n - \left(x - \frac{1}{x}\right)^n \right] u\right) \\ &= \frac{1}{4}x^2 - \frac{1}{4} \left[\left(x + \frac{1}{x}\right)^n - \left(x - \frac{1}{x}\right)^n \right]^2 = B^n, \end{aligned} \quad (10)$$

where we used (5).

Thus,

$$\left(x + \frac{1}{x}\right)^n - \left(x - \frac{1}{x}\right)^n = \sqrt{x^2 - 4B^n}. \quad (11)$$

Solving (2b) for x^2 , we get

$$x^2 = \frac{B + \sqrt{B^2 + 4}}{2}. \quad (12)$$

So, finally, the answer is

$$\left(x + \frac{1}{x}\right)^n - \left(x - \frac{1}{x}\right)^n = \sqrt{\frac{B + \sqrt{B^2 + 4}}{2} - 4B^n}. \quad (13)$$