Olympiad Problem 12

P. Reany

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Abstract

Here we use the unipodal algebra to assist in solving the problem, which is given to us on YouTube. Although I'm referring to the series under the name 'olympiad', the problems are from diverse sources as olympiads, entrance exams, SATs, and the like.

The greatest killer of creativity is interruption.
— John Cleese

Mathematics is the art of reducing any
problem to linear algebra.
— William Stein

The YouTube video is found at:

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https://www.youtube.com/watch?v=Iq-ToVDngHA
 Titled: A nice Math Olympiad Problem || Find x=? & y=?
 Presenter: Super Academy
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1 The Problem

Given the relations

$$\sqrt{x} + \sqrt{y} = 5\,, \tag{1}$$

$$\sqrt{x+16} - \sqrt{y+5} = 2\,, \tag{2}$$

solve for x and y over the real numbers.

2 The Prerequisites: The unipodal algebra

This algebra is formed as the extension of the complex numbers by the number u, where $u^2 = 1$, and u commutes with the complex numbers. The number u is said to be 'unipotent'. The set of numbers constructed this way are the unipodal numbers, a particular such number is called a unipode. The main conjugation operator on unipode a is the unegation operator, written a^-. It does not affect complex numbers, but it sends every u to its negative. Hence, if $a = x + yu$, where x, y are complex numbers, then $a^- = x - yu$. Unegation distributes over addition and multiplication.

The following are some properties that will come in handy:

$$u^2 = 1\,, \tag{3a}$$

$$u_\pm \equiv \tfrac{1}{2}(1 \pm u)\,, \tag{3b}$$

$$u_\pm^2 = u_\pm\,, \tag{3c}$$

$$u = u_+ - u_-\,, \tag{3d}$$

$$u_+ u_- = 0\,, \tag{3e}$$

$$u_+ + u_- = 1\,, \tag{3f}$$

$$u u_+ = u_+\,, \tag{3g}$$

$$u u_- = -u_-\,, \tag{3h}$$

$$(u_\pm)^- = u_\mp\,. \tag{3i}$$

You should prove (3c) – (3i). By the way, these two special unipodes $u_\pm$ square to themselves. Such numbers in a ring are referred to as *idempotents*. In the unipodal numbers they have no inverses. The fact that the unipodal number system is not a field is of little concern to me. In fact, most unipodes have inverses, so long as they are not multiples of one of the idempotents. If one needs field elements, the scalars of the unipodal numbers comprise the field of complex numbers.

Two often-used results are, for complex numbers w, z (which are used to convert unipodes between the bases $\{1, u\}$ and $\{u_+, u_-\}$):

$$w + zu = (w + z)u_+ + (w - z)u_-\,, \tag{4a}$$

$$wu_+ + zu_- = \tfrac{1}{2}(w + z) + \tfrac{1}{2}(w - z)u\,. \tag{4b}$$

The unipodal algebra has two copies of the complex numbers, one for each component. In any true unipodal equation, the corresponding coefficients across the equal sign are equal to each other. This is similar to equating real and imaginary components across the equal sign in the complex algebra.

When I first used the unipodal algebra to solve polynomial equations (c. 1984-5), I used the Clifford 1 algebra over the complex numbers. The '1' means one unit vector u. So, a Clifford 1 number c can be represented as

$$c = a + bu\,, \tag{5}$$

where a, b are complex numbers. Of course, u being a unit vector, then

$$u^2 = 1\,. \tag{6}$$

Now, the standard basis for this space is $\{1, u\}$ and the scalars are the complex numbers. To extract the 'scalar part' of (5), we use the selection operator $\langle\,\cdot\,\rangle$, as follows:

$$\langle\, c\,\rangle = \langle\, a + bu\,\rangle = a\,, \tag{7}$$

One can also subscript the selector with a zero for the scalar part:

$$\langle\, c \,\rangle_0 = \langle\, a + bu \,\rangle_0 = a\,, \tag{8}$$

and with a '1' for the vector part:

$$\langle\, c \,\rangle_1 = \langle\, a + bu \,\rangle_1 = bu\,. \tag{9}$$

When I adopted the name 'unipodal algebra' from a paper I cowrote with two other authors, I found a need to adopt new terminology for naming the scalar and vector parts. Just as complex numbers are composed of a real number times the unit '1' and another real number times the unit imaginary i, the unipodal numbers are composed of a complex number times the unit '1' and another complex number times the unipotent number u. The part of the unipode that does not contain the unipotent factor is called the 'complex part' of the unipode. The part that does contain the unipotent element factor is called the **uniplex part** of the unipode.

Now, before you complain that calling the scalar part of a unipode the 'complex part' is nonsense, I point out that in complex analysis, the nonimaginary part is referred to as the 'real part'. Lastly, when I say the 'uniplex part' in this series of papers, I refer only to the coefficient of the nonscalar part, which is complex only. Thus the uniplex part of unipode $c = a + bu$ is just b. Another way to think of the uniplex part of c is to take the scalar (or complex) part of cu.

$$\langle\, c \,\rangle_1 = \langle\, cu \,\rangle = \langle\, au + b \,\rangle = b\,. \tag{10}$$

Thus, one must be careful when I report I'm taking the uniplex part of a unipode (across all the papers I've written over the years), because at times it may contain that factor of u and at other times not. But like I said: In this series it will always mean only the scalar factor of the unipotent element.

Much of the algebraic power of the unipodal algebra comes from 1) it being able to switch the presentation of a unipode between the standard basis and the idempotent basis, the latter basis being well suited for taking powers and roots.

3 The First Solution

I'll begin by making the standard constructions.

$$\sqrt{x} - \sqrt{y} = 2k\,, \tag{11}$$

$$\sqrt{x+16} + \sqrt{y+5} = 2\ell\,, \tag{12}$$

This time, I'll introduce two 'first unipodes', one for each given constraint:

$$a = \sqrt{x+16}\,u_+ + \sqrt{y+5}\,u_-\,, \tag{13}$$

$$b = \sqrt{x}\,u_+ + \sqrt{y}\,u_-\,. \tag{14}$$

Let's begin by squaring both unipodes so we can look for some easy relationship between them:

$$a^2 = (x+16)\,u_+ + (y+5)\,u_- \,, \tag{15}$$
$$b^2 = x\,u_+ + y\,u_- \,. \tag{16}$$

So, one relationship stands out:

$$a^2 = b^2 + 16\,u_+ + 5\,u_- \,. \tag{17}$$

Next, we have to put a and b in standard forms to use the given constraints:

$$\begin{aligned} a &= \tfrac{1}{2}[\sqrt{x+16} + \sqrt{y+5}] + \tfrac{1}{2}[\sqrt{x+16} - \sqrt{y+5}]\,u \\ &= \ell + u\,. \end{aligned} \tag{18}$$
$$a^2 = (\ell^2 + 1) + 2\ell u\,. \tag{19}$$

If we go back to (15) and put it in standard form, we get

$$a^2 = \tfrac{1}{2}[(x+16) + (y+5)] + \tfrac{1}{2}[(x+16) - (y+5)]\,u \tag{20}$$
$$= \tfrac{1}{2}[x+y+21] + \tfrac{1}{2}[x-y+11]\,u\,. \tag{21}$$

On comparing (19) to (21), we have that

$$x + y = 2\ell^2 - 19\,, \tag{22}$$
$$x - y = 4\ell - 11\,. \tag{23}$$

So, from these, if we can solve for ℓ, we can quickly solve for x and y:

$$x = \ell^2 + 2\ell - 15\,, \tag{24}$$
$$y = \ell^2 - 2\ell - 4\,. \tag{25}$$

What we did to a we need to do likewise to b.

$$\begin{aligned} b &= \tfrac{1}{2}[\sqrt{x} + \sqrt{y}] + \tfrac{1}{2}[\sqrt{x} - \sqrt{y}]\,u \\ &= \frac{5}{2} + ku\,. \end{aligned} \tag{26}$$
$$b^2 = \left(\frac{25}{4} + k^2\right) + 5ku\,. \tag{27}$$

On placing a^2 from (19) and b^2 from (27) into (17), we get

$$(\ell^2 + 1) + 2\ell u = \left(\frac{25}{4} + k^2\right) + 5ku + \tfrac{1}{2}(16+5) + \tfrac{1}{2}(16-5)u\,. \tag{28}$$

Next, we equate complex and uniplex parts:

$$\ell^2 + 1 = \left(\frac{25}{4} + k^2\right) + \frac{21}{2}\,, \tag{29}$$

$$2\ell = 5k + \frac{11}{2}\,. \tag{30}$$

WolframAlpha.com provides the following solutions:

$$k = \frac{1}{2}, \quad \ell = 4\,, \tag{31}$$

$$k = -\frac{131}{42}, \quad \ell = -\frac{106}{21}\,. \tag{32}$$

From (26),

$$\begin{aligned} b &= \tfrac{1}{2}[\sqrt{x} + \sqrt{y}] + \tfrac{1}{2}[\sqrt{x} - \sqrt{y}]\,u \\ &= \frac{5}{2} + ku = \Big(\frac{5}{2} + k\Big)u_+ + \Big(\frac{5}{2} - k\Big)u_-\,. \end{aligned} \tag{33}$$

$$\begin{aligned} b^2 &= xu_+ + yu_- \\ &= \tfrac{1}{2}[x + y] + \tfrac{1}{2}[x - y]\,u\,. \end{aligned} \tag{34}$$

For case $k = \frac{1}{2}$, $\ell = 4$, we use (24) and (25):

$$x = 9\,, \tag{35}$$

$$y = 4\,. \tag{36}$$

4 The Second Solution

Is this second solution really better than the first, or is it even useful at all? Well, I feel like a pioneer who is exploring a region of mathematics that is, at this point, not well trodden. To me, what's important are the techniques used, not the answer, per se. This time around, I'm using the technique of 'pivoting', which I didn't conceive of until later in the series. I'm also trying to establish *pure unipodes* in order to efficiently use them, if I can.[1] Only time will tell if this approach has merit.

One of the experimental techniques I'm trying out in this problem is to define more than one initial unipodes and to find a way to combine them productively. I think that that much was accomplished here.

[1] A 'pure unipode' is a unipode whose components are, at best, just numbers, or, at worst, may also be parameters that are not dependent on the variables of the problem.

Now, I'll introduce two 'first unipodes', one for each given relation, beginning with the first Given relation:

$$a = \sqrt{x}\,u_+ + \sqrt{y}\,u_- \tag{37a}$$
$$= \tfrac{1}{2}(\sqrt{x} + \sqrt{y}) + \tfrac{1}{2}(\sqrt{x} - \sqrt{y})\,u \tag{37b}$$
$$= \frac{5}{2} + \frac{k}{2}\,u\,, \tag{37c}$$
$$a^2 = x\,u_+ + y\,u_-\,, \tag{37d}$$

where we used (1) and defined k as:

$$k \equiv \sqrt{x} - \sqrt{y}\,. \tag{38}$$

Now, for the second Given relation:

$$b = \sqrt{x+16}\,u_+ - \sqrt{y+5}\,u_- \tag{39a}$$
$$= \tfrac{1}{2}(\sqrt{x+16} - \sqrt{y+5}) + \tfrac{1}{2}(\sqrt{x+16} + \sqrt{y+5})\,u \tag{39b}$$
$$= 1 + \frac{\ell}{2}\,u\,, \tag{39c}$$
$$b^2 = (x+16)\,u_+ + (y+5)\,u_-\,, \tag{39d}$$

where we used (2) and defined ℓ as:

$$\ell \equiv \sqrt{x+16} + \sqrt{y+5}\,. \tag{40}$$

(The status of ℓ is somewhat of a placeholder, as we will get what we need by solving for k alone.)

Taking the difference of the squares of the unipodes gives us a pure unipode:

$$b^2 - a^2 = 16\,u_+ + 5\,u_-\,. \tag{41}$$

But can we make good use of it? Let's pivot on both components, using (37c) and (39c) as their commodities:[2]

$$16 = \langle\, b^2 - a^2 \,\rangle_{u_+} = \langle\, \big(1 + \frac{\ell}{2}u\big)^2 - \big(\frac{5}{2} + \frac{k}{2}u\big)^2 \,\rangle_{u_+} \tag{42a}$$
$$= \langle\, \big(1 + \frac{\ell}{2}u\big)^2 u_+ - \big(\frac{5}{2} + \frac{k}{2}u\big)^2 u_+ \,\rangle_{u_+} \tag{42b}$$
$$= \big(1 + \frac{\ell}{2}\big)^2 - \big(\frac{5}{2} + \frac{k}{2}\big)^2\,. \tag{42c}$$

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[2]I'd say that we made good use of this pure unipode, because it gave us two pure numbers to pivot on.

Likewise, for the other component:

$$5 = \langle\, b^2 - a^2 \,\rangle_{u_-} = \langle\, \big(1 + \frac{\ell}{2} u\big)^2 - \big(\frac{5}{2} + \frac{k}{2} u\big)^2 \,\rangle_{u_-} \tag{43a}$$

$$= \langle\, \big(1 + \frac{\ell}{2} u\big)^2 u_- - \big(\frac{5}{2} + \frac{k}{2} u\big)^2 u_- \,\rangle_{u_-} \tag{43b}$$

$$= \big(1 + \frac{\ell}{2}\big)^2 - \big(\frac{5}{2} - \frac{k}{2}\big)^2 . \tag{43c}$$

On eliminating ℓ between (42c) and (43c) (remembering to include plus and minus sign when taking square roots), we have that[3]

$$2 = \sqrt{\big(16 + \big(\frac{5}{2} + \frac{k}{2}\big)^2} - \sqrt{\big(5 + \big(\frac{5}{2} + \frac{k}{2}\big)^2} \,, \tag{44a}$$

$$2 = -\sqrt{\big(16 + \big(\frac{5}{2} + \frac{k}{2}\big)^2} + \sqrt{\big(5 + \big(\frac{5}{2} + \frac{k}{2}\big)^2} \,. \tag{44b}$$

WolframAlpha tells me that the solution for k for (44a) is $k = 1$, and for (44b) is $k = -131/21$.

We can now easily solve for x and y by using $k = 1$ in (38) and coupling that to (1), to get

$$x = 9 \,, \tag{45a}$$

$$y = 4 \,. \tag{45b}$$

So, was it worth all this effort to arrive at (44a)? I suppose that depends on one's point of view. There's always the standard methods if you don't want to use the unipodal methods. But in its favor, we were given initially two coupled equations in two unknowns, and this procedure allowed us to reduce that to one equation in one unknown. Besides all that, I may have not found the most efficient solution. Time will tell.

As a final comment, I'd like to address a possible misconception I may have engendered. Although I have made a big deal out of pivoting on the components of pure unipodes, one can also pivot on the components of non-pure unipodes. I have a vague memory of having done so successfully in one of the other problems in this list.

[3]When I used only plus signs, WolframAlpha told me that there was no solution.