Olympiad Problem 13

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Abstract

Here we use the unipodal algebra to assist in solving the problem, which is given to us on YouTube. Although I'm referring to the series under the name 'olympiad', the problems are from diverse sources as olympiads, entrance exams, SATs, and the like.

> You have to know what to look for, so you can spot it.
> — Papago Indian drug-enforcement border scout

The YouTube video is found at:

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https://www.youtube.com/watch?v=v_QyUSa-_ME
 Titled: Math Olympiad Problem | A Nice Algebra Challenge
 Presenter: Math Booster
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1 The Problem

Note: There are many other presenters for this problem on YouTube.

Given the relations

$$a + b = 1\,, \tag{1a}$$
$$a^2 + b^2 = 2\,, \tag{1b}$$

find the value of $a^{11} + b^{11}$.

2 The Prerequisites: The unipodal algebra

This algebra is formed as the extension of the complex numbers by the number u, where $u^2 = 1$, and u commutes with the complex numbers. The number u is said to be 'unipotent'. The set of numbers constructed this way are the unipodal numbers, a particular such number is called a unipode. The main conjugation

operator on unipode a is the unegation operator, written a^-. It does not affect complex numbers, but it sends every u to its negative. Hence, if $a = x + yu$, where x, y are complex numbers, then $a^- = x - yu$. Unegation distributes over addition and multiplication.

The following are some properties that will come in handy:

$$u^2 = 1\,, \tag{2a}$$
$$u_\pm \equiv \tfrac{1}{2}(1 \pm u)\,, \tag{2b}$$
$$u_\pm^2 = u_\pm\,, \tag{2c}$$
$$u = u_+ - u_-\,, \tag{2d}$$
$$u_+ u_- = 0\,, \tag{2e}$$
$$u_+ + u_- = 1\,, \tag{2f}$$
$$u u_+ = u_+\,, \tag{2g}$$
$$u u_- = -u_-\,, \tag{2h}$$
$$(u_\pm)^- = u_\mp\,. \tag{2i}$$

You should prove (2c) – (2i). By the way, these two special unipodes $u_\pm$ square to themselves. Such numbers in a ring are referred to as *idempotents*. In the unipodal numbers they have no inverses. The fact that the unipodal number system is not a field is of little concern to me. In fact, most unipodes have inverses, so long as they are not multiples of one of the idempotents. If one needs field elements, the scalars of the unipodal numbers comprise the field of complex numbers.

Two often-used results are, for complex numbers w, z (which are used to convert unipodes between the bases $\{1, u\}$ and $\{u_+, u_-\}$):

$$w + zu = (w + z)u_+ + (w - z)u_-\,, \tag{3a}$$
$$w u_+ + z u_- = \tfrac{1}{2}(w + z) + \tfrac{1}{2}(w - z)u\,. \tag{3b}$$

The unipodal algebra has two copies of the complex numbers, one for each component. In any true unipodal equation, the corresponding coefficients across the equal sign are equal to each other. This is similar to equating real and imaginary components across the equal sign in the complex algebra.

When I first used the unipodal algebra to solve polynomial equations (c. 1984-5), I used the Clifford 1 algebra over the complex numbers. The '1' means one unit vector u. So, a Clifford 1 number c can be represented as

$$c = a + bu\,, \tag{4}$$

where a, b are complex numbers. Of course, u being a unit vector, then

$$u^2 = 1\,. \tag{5}$$

Now, the standard basis for this space is $\{1, u\}$ and the scalars are the complex numbers. To extract the 'scalar part' of (4), we use the selection operator $\langle\,\cdot\,\rangle$,

as follows:

$$\langle\, c \,\rangle = \langle\, a + bu \,\rangle = a\,, \tag{6}$$

One can also subscript the selector with a zero for the scalar part:

$$\langle\, c \,\rangle_0 = \langle\, a + bu \,\rangle_0 = a\,, \tag{7}$$

and with a '1' for the vector part:

$$\langle\, c \,\rangle_1 = \langle\, a + bu \,\rangle_1 = bu\,. \tag{8}$$

When I adopted the name 'unipodal algebra' from a paper I cowrote with two other authors, I found a need to adopt new terminology for naming the scalar and vector parts. Just as complex numbers are composed of a real number times the unit '1' and another real number times the unit imaginary i, the unipodal numbers are composed of a complex number times the unit '1' and another complex number times the unipotent number u. The part of the unipode that does not contain the unipotent factor is called the 'complex part' of the unipode. The part that does contain the unipotent element factor is called the **uniplex part** of the unipode.

Now, before you complain that calling the scalar part of a unipode the 'complex part' is nonsense, I point out that in complex analysis, the nonimaginary part is referred to as the 'real part'. Lastly, when I say the 'uniplex part' in this series of papers, I refer only to the coefficient of the nonscalar part, which is complex only. Thus the uniplex part of unipode $c = a + bu$ is just b. Another way to think of the uniplex part of c is to take the scalar (or complex) part of cu.

$$\langle\, c \,\rangle_1 = \langle\, cu \,\rangle = \langle\, au + b \,\rangle = b\,. \tag{9}$$

Thus, one must be careful when I report I'm taking the uniplex part of a unipode (across all the papers I've written over the years), because at times it may contain that factor of u and at other times not. But like I said: In this series it will always mean only the scalar factor of the unipotent element.

Much of the algebraic power of the unipodal algebra comes from 1) it being able to switch the presentation of a unipode between the standard basis and the idempotent basis, the latter basis being well suited for taking powers and roots.

3 The Solution

I'll begin by making the standard constructions. Let

$$a - b = 2k\,. \tag{10}$$

So, from this last equation and (1a), we have that

$$a^2 - b^2 = 2k\,. \tag{11}$$

This time, I'll introduce the 'first unipode' X as:

$$X = a u_+ + b u_- \,, \tag{12a}$$
$$= \tfrac{1}{2}(a+b) + \tfrac{1}{2}(a-b)u = \tfrac{1}{2} + ku \,. \tag{12b}$$

Let's get a corresponding couple of equations for X^2:

$$X^2 = a^2 u_+ + b^2 u_- \,, \tag{13a}$$
$$= \tfrac{1}{2}(a^2+b^2) + \tfrac{1}{2}(a^2-b^2)u = 1 + ku \,, \tag{13b}$$

where we used (1b) and (11).

Now, at this point of the series of papers on the unipodal algebra to solve Olympiad problems has it gotten especially exciting, because for the first time we can write a nontrivial polynomial in the unipode X. Here's how it works. Notice that if we add $\frac{1}{2}$ to X in (12b), we get X^2 in (13b)! The equational form we'll use is

$$X^2 = X + \tfrac{1}{2} \,. \tag{14}$$

So, before this, we took roots and powers of unipodes and that took us far, but this will be different.

To get to $a^{11} + b^{11}$ all we need to do is to get to X^{11}, but to do so efficiently. So, let's get started (I'm going to leave out a lot of the boring arithmetic)

$$X^4 = (X^2)^2 = (X + \tfrac{1}{2})^2 = 2X + \frac{3}{4} \,. \tag{15}$$

And then,

$$X^8 = (X^4)^2 = (2X + \frac{3}{4})^2 = 7X + \frac{41}{16} \,. \tag{16}$$

And then,

$$X^{10} = X^2 \cdot X^8 = (X + \tfrac{1}{2})\big(7X + \frac{41}{16}\big) = \big(\frac{418}{32}X + \frac{153}{32}\big) \,. \tag{17}$$

And then,

$$X^{11} = X \cdot X^{10} = X\big(\frac{418}{32}X + \frac{153}{32}\big) = \frac{571}{32}X + \frac{209}{32} \,. \tag{18}$$

Well, we're almost there. All we need to do now is to convert to idempotent form and add the coefficients on both sides.

$$a^{11} u_+ + b^{11} u_- = \frac{571}{32}(a u_+ + b u_-) + \frac{209}{32}(u_+ + u_-) \,. \tag{19}$$

Therefore,

$$a^{11} = \frac{571}{32} a + \frac{209}{32} \,, \tag{20a}$$
$$b^{11} = \frac{571}{32} b + \frac{209}{32} \,. \tag{20b}$$

Now we just add them together:

$$a^{11} + b^{11} = \frac{571}{32}(a+b) + \frac{418}{32} = \frac{989}{32}\,, \tag{21}$$

where we used (1a). By the way, this is the same answer that Mathematica provided me, when it did the computations however it decided to do them.

4 Conclusion

I hope that was as much 'fun' for you as it was for me. Trust me on this, there's a lot of room to make plenty of arithmetic mistakes on the computations I omitted. I left them out, of course, because those steps are ugly.

It's interesting to note that in this solution we never needed to solve for k, which we usually did in all the previous problems.

Lastly, it occurs to me that there ought to exist some simple recursion formula that would skip the tedious iteration of steps. Something like this

$$X^n = f(n-1)X + g(n-1)\,. \tag{22}$$

But I've accomplished enough for this problem for now, which is the proof of concept that employing polynomials in the unipodal variables can be a useful strategy in problem-solving.