Olympiad Problem 14

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Abstract

Here we use the unipodal algebra to assist in solving the problem, which is given to us on YouTube. Although I'm referring to the series under the name 'olympiad', the problems are from diverse sources as olympiads, entrance exams, SATs, and the like.

The YouTube video is found at:

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https://www.youtube.com/watch?v=-B59K4zuyGo
  Titled: Harvard University Aptitude Test Strategy You
       Didn't Know Existed || Algebra Problem || 99% Failed
  Presenter: Super Academy
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1 The Problem

People often overlook the obvious.
— Doctor Who

Given the relations

$$a^2 = b + 183\,, \tag{1a}$$

$$b^2 = a + 183\,, \tag{1b}$$

where $a \neq b$, find the values of a and b.

2 The Prerequisites: The unipodal algebra

This algebra is formed as the extension of the complex numbers by the number u, where $u^2 = 1$, and u commutes with the complex numbers. The number u is said to be 'unipotent'. The set of numbers constructed this way are the unipodal numbers, a particular such number is called a unipode. The main conjugation operator on unipode a is the unegation operator, written a^-. It does not affect complex numbers, but it sends every u to its negative. Hence, if $a = x + yu$, where x, y are complex numbers, then $a^- = x - yu$. Unegation distributes over addition and multiplication.

The following are some properties that will come in handy:

$$u^2 = 1\,, \tag{2a}$$
$$u_\pm \equiv \tfrac{1}{2}(1 \pm u)\,, \tag{2b}$$
$$u_\pm^2 = u_\pm\,, \tag{2c}$$
$$u = u_+ - u_-\,, \tag{2d}$$
$$u_+ u_- = 0\,, \tag{2e}$$
$$u_+ + u_- = 1\,, \tag{2f}$$
$$uu_+ = u_+\,, \tag{2g}$$
$$uu_- = -u_-\,, \tag{2h}$$
$$(u_\pm)^- = u_\mp\,. \tag{2i}$$

You should prove (2c) – (2i). By the way, these two special unipodes $u_\pm$ square to themselves. Such numbers in a ring are referred to as *idempotents*. In the unipodal numbers they have no inverses. The fact that the unipodal number system is not a field is of little concern to me. In fact, most unipodes have inverses, so long as they are not multiples of one of the idempotents. If one needs field elements, the scalars of the unipodal numbers comprise the field of complex numbers.

Two often-used results are, for complex numbers w, z (which are used to convert unipodes between the bases $\{1, u\}$ and $\{u_+, u_-\}$):

$$w + zu = (w + z)u_+ + (w - z)u_-\,, \tag{3a}$$
$$wu_+ + zu_- = \tfrac{1}{2}(w + z) + \tfrac{1}{2}(w - z)u\,. \tag{3b}$$

The unipodal algebra has two copies of the complex numbers, one for each component. In any true unipodal equation, the corresponding coefficients across the equal sign are equal to each other (when they're in the same basis). This is similar to equating real and imaginary components across the equal sign in the complex algebra.

Next, we learn how to take the 'norm' of a unipode. Let w be a unipode in standard basis, given by

$$w = a + bu\,, \tag{4}$$

where a, b are complex numbers. The 'norm' of w is given as

$$ww^- = (a + bu)(a - bu) = a^2 - b^2\,. \tag{5}$$

Now, let y be a unipode in idempotent basis, given as

$$y = Au_+ + Bu_-\,, \tag{6}$$

where A, B are complex numbers. The 'norm' of y is given as

$$\begin{aligned} yy^- &= (Au_+ + Bu_-)(Au_- + Bu_+) = ABu_+ + ABu_- \\ &= AB(u_+ + u_-) = AB\,. \end{aligned} \tag{7}$$

When I first used the unipodal algebra to solve polynomial equations (c. 1984-5), I used the Clifford 1 algebra over the complex numbers. The '1' means one unit vector u. So, a Clifford 1 number c can be represented as

$$c = a + bu\,, \tag{8}$$

where a, b are complex numbers. Of course, u being a unit vector, then

$$u^2 = 1\,. \tag{9}$$

Now, the standard basis for this space is $\{1, u\}$ and the scalars are the complex numbers. To extract the 'scalar part' of (8), we use the selection operator $\langle\,\cdot\,\rangle$, as follows:

$$\langle\, c\,\rangle = \langle\, a + bu\,\rangle = a\,, \tag{10}$$

One can also subscript the selector with a zero for the scalar part:

$$\langle\, c\,\rangle_0 = \langle\, a + bu\,\rangle_0 = a\,, \tag{11}$$

and with a '1' for the vector part:

$$\langle\, c\,\rangle_1 = \langle\, a + bu\,\rangle_1 = bu\,. \tag{12}$$

When I adopted the name 'unipodal algebra' from a paper I cowrote with two other authors, I found a need to adopt new terminology for naming the scalar and vector parts. Just as complex numbers are composed of a real number times the unit '1' and another real number times the unit imaginary i, the unipodal numbers are composed of a complex number times the unit '1' and another complex number times the unipotent number u. The part of the unipode that does not contain the unipotent factor is called the 'complex part' of the unipode. The part that does contain the unipotent element factor is called the **uniplex part** of the unipode.

Now, before you complain that calling the scalar part of a unipode the 'complex part' is nonsense, I point out that in complex analysis, the nonimaginary part is referred to as the 'real part'. Lastly, when I say the 'uniplex part' in this series of papers, I refer only to the coefficient of the nonscalar part, which is complex only. Thus the uniplex part of unipode $c = a + bu$ is just b. Another way to think of the uniplex part of c is to take the scalar (or complex) part of cu.

$$\langle\, c\,\rangle_1 = \langle\, cu\,\rangle = \langle\, au + b\,\rangle = b\,. \tag{13}$$

Thus, one must be careful when I report I'm taking the uniplex part of a unipode (across all the papers I've written over the years), because at times it may contain that factor of u and at other times not. But like I said: In this series it will always mean only the scalar factor of the unipotent element.

Much of the algebraic power of the unipodal algebra comes from 1) it being able to switch the presentation of a unipode between the standard basis and the idempotent basis, the latter basis being well suited for taking powers and roots.

3 The Solution

Let's begin by exploring the consequences of this stipulation that $a \neq b$. On subtracting (1b) from (1a), we have that

$$a^2 - b^2 = b - a\,. \tag{14}$$

Now, since $a \neq b$, then $a - b \neq 0$, and we can divide through by it. Thus,

$$a + b = -1\,. \tag{15}$$

Let's not miss the possible importance of this last relation: First, because it's on a and b, and, second, because it's simple. [1]

I'll begin by making the standard construction. Let

$$X = au_+ + bu_- = \tfrac{1}{2}(a+b) + \tfrac{1}{2}(a-b)u \tag{16a}$$
$$= -\tfrac{1}{2} + \tfrac{1}{2}(a-b)u \tag{16b}$$
$$= (-\tfrac{1}{2} + \tfrac{1}{2}(a+b))u_+ + (-\tfrac{1}{2} - \tfrac{1}{2}(a-b))u_-\,, \tag{16c}$$

where we used (15) in (16a) to get to the next line. On equating the components of the first line to the last line:

$$a = -\tfrac{1}{2} + \tfrac{1}{2}(a+b)\,, \tag{17a}$$
$$b = -\tfrac{1}{2} - \tfrac{1}{2}(a-b)\,, \tag{17b}$$

which become

$$a = -1 - b\,, \tag{18a}$$
$$b = -1 - a\,. \tag{18b}$$

By substituting (18a) into (1a) we get

$$a^2 + a - 182 = 0\,, \tag{19}$$

which has roots $a = 13, -14$. Now, because the equations hold their form under interchange of a and b, then the solution set for a and b are given as

$$(a, b) = \{(13, -14), (-14, 13)\}\,. \tag{20}$$

[1] We can skip using the unipodal algebra at this point by solving (15) for b and substituting that into (1a). The unipodal approach will amount to the same thing, but with more steps. Fact is, I saw this shorter proof after I wrote up the unipodal proof.

4 Conclusion

Let's take a moment to briefly take stock of the unipodal techniques we've used so far in this series that have been useful (and add in one or two that might be useful in the future):

1) Forming the 'first unipode' wisely.

2) Taking roots or powers, especially on unipodes in the idempotent basis.

3) 'Flipping' between bases.

4) Extracting the complex and/or uniplex parts across an equation.

5) Taking the 'magnitude square' of a unipode. For example, if $X = x_0 + x_1 u$, $XX^- = x_0^2 - x_1^2$, which is, of course, just a complex number. If two unipodes are equal, their square magnitudes are equal, and you are free to calculate their square magnitudes from either basis.

6) Comparing square magnitudes this way: $X^n (X^-)^n = (XX^-)^n$.

7) If A and B are equal unipodes in standard form, then $\frac{a_0}{a_1} = \frac{b_0}{b_1}$, but if they are in idempotent form, then $\frac{a_+}{a_-} = \frac{b_+}{b_-}$. Taking the ratio of components can be particularly useful when the components have a nontrivial common factor — it will be divided out, of course, and that could lead to a simplification.

8) If A and B are equal unipodes in standard form ($A = a_0 + a_1 u$, $B = b_0 + b_1 u$) then $a_0 a_1 = b_0 b_1$, but if they are in idempotent form ($A = a_+ u_+ + a_- u_-$, $B = b_+ u_+ + b_- u_-$) , then $a_+ a_- = b_+ b_-$.

The features described in 7) and 8) provide us with quick ways to obtain scalar information out of unipodal equations.

Furthermore, we can add to these tricks all the techniques of real and complex number and ring theory.