

Olympiad Problem 145

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I love it when a plan comes together.

— Hannibal Smith, *The A-Team*

The YouTube video is found at:

Source: <https://www.youtube.com/watch?v=wgd6Gp-Kfbk>

Title: This Question Tricked Thousands of Students

Presenter: J Educational Tutorials

1 The Problem

Given the relation

$$x^2 - 5x + 7 = 0, \quad (1)$$

find the value of ϕ given by

$$\phi \equiv (x - 2)^{90} + (3 - x)^{90}. \quad (2)$$

2 The Solution

My first job is to rewrite ϕ as

$$\phi = (x - 2)^{90} + (x - 3)^{90}. \quad (3)$$

So, I chose as my 'first unipode' and follow-up:

$$a = (x - 2)u_+ + (x - 3)u_- \quad (4a)$$

$$= \frac{1}{2}[(x - 2) + (x - 3)] + \frac{1}{2}[(x - 2) - (x - 3)]u \quad (4b)$$

$$= \left(x - \frac{5}{2}\right) + \frac{1}{2}u. \quad (4c)$$

$$a^{90} = (x - 2)^{90}u_+ + (x - 3)^{90}u_- \quad (4d)$$

$$= \frac{1}{2}[(x - 2)^{90} + (x - 3)^{90}] + \frac{1}{2}[(x - 2)^{90} - (x - 3)^{90}]u \quad (4e)$$

$$= \frac{1}{2}\phi + \frac{1}{2}\psi u, \quad (4f)$$

where we defined

$$\psi = (x - 2)^{90} - (x - 3)^{90}. \quad (5)$$

Now, if we can find a numerical value for $x - 5/2$ in (4c), we can reduce a to a pure unipode. If we can complete the square on (1), we can solve for $(x - 5/2)^2$

$$(x - 5/2)^2 = -\frac{3}{4}. \quad (6)$$

On taking the square root on both sides, we have that

$$x - 5/2 = \pm i \frac{\sqrt{3}}{2}. \quad (7)$$

Therefore

$$a = \pm i \frac{\sqrt{3}}{2} + \frac{1}{2}u. \quad (8)$$

Now for a little algebraic sleight of hand, multiply through by u and reorder terms on the right:

$$au = \frac{1}{2} \pm I \frac{\sqrt{3}}{2} = e^{\pm I\pi/3}, \quad (9)$$

where we have introduced I as

$$I \equiv ui, \quad (10)$$

and

$$I^2 = (ui)^2 = -1. \quad (11)$$

In other words, I acts just like a unit imaginary.

So,

$$a^{90} = (au)^{90} = (e^{\pm I\pi})^{30} = (-1)^{30} = 1. \quad (12)$$

On comparing this last result with (4f), we conclude that

$$\phi = 2. \quad (13)$$