Olympiad Problem 147

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If there is to be a brave new world, our generation is
going to have the hardest time living in it.
— Chancellor Gorkon (Star Trek VI,
The Undiscovered Country)

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Source: https://www.youtube.com/watch?v=aX4IHM0VcXk
 Title: A fancier way to solve this radical equation
 Presenter: blackpenredpen
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1 The Problem

Given the relation

$$\sqrt[3]{x-40} + \sqrt[3]{-x+3} = -1\,, \tag{1}$$

find the real values of x.

2 The Solution

So, I chose as my 'first unipode' and follow-up:

$$\begin{aligned} a &= \sqrt[3]{x-40}\,u_+ + \sqrt[3]{-x+3}\,u_- && \text{(2a)} \\ &= \frac{1}{2}\big[\sqrt[3]{x-40} + \sqrt[3]{-x+3}\big] + \frac{1}{2}\big[\sqrt[3]{x-40} - \sqrt[3]{-x+3}\big]\,u && \text{(2b)} \\ &= -\frac{1}{2} + \frac{1}{2}B\,u\,, && \text{(2c)} \end{aligned}$$

where

$$B \equiv \sqrt[3]{x-40} - \sqrt[3]{-x+3}\,. \tag{3}$$

On cubing a in (2c), and then taking its scalar part, we have that

$$\langle\, a^3 \,\rangle_0 = (-\tfrac{1}{2})^3 + 3(-\tfrac{1}{2})B^2/4\,. \tag{4}$$

On cubing a in (2a), and then converting to its scalar and unipotent parts, we have that

$$a^3 = (x-40)\,u_+ + (-x+3)\,u_- \tag{5a}$$

$$= \frac{1}{2}(-37) + (\text{unipotent part})\,u\,. \tag{5b}$$

Now, the scalar part of this last equation must equal the expression in (4), hence

$$\frac{1}{2}(-37) = (-\tfrac{1}{2})^3 + 3(-\tfrac{1}{2})B^2/4\,. \tag{6}$$

This equation gives us two values for B:

$$B = +7, \quad -7\,. \tag{7}$$

I think that the simplest way to proceed is to add equations (1) and (3) together, to get (after a lot of algebra), for $B = +7$

$$x = 67\,. \tag{8}$$

And for $B = -7$, we get

$$x = -24\,. \tag{9}$$