Olympiad Problem 148

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The ugliest curve you're ever likely to see in your life
is the *learning curve*. Deal with it.
— The Author

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Source: https://www.youtube.com/watch?v=euBnlvAanxY
Title: France | Can you solve?
 Presenter: Math Master TV
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1 Problem

The following relation

$$\phi = (\sqrt{11} + \sqrt{5}\,)^8 + (\sqrt{11} - \sqrt{5}\,)^8 \,, \tag{1}$$

has a simple (though large) integer value. Find it.

2 Solution

This can be solved in the unipodal algebra, so I'll do that just to see how the solution goes.

For my initial unipodal element, I'll start with

$$a = (\sqrt{11} + \sqrt{5}\,)\, u_+ + (\sqrt{11} - \sqrt{5}\,)\, u_+ \,, \tag{2}$$

which is in the idempotent basis. Converted to the standard basis, we have that

$$a = \sqrt{11} + \sqrt{5}\, u \,. \tag{3}$$

Now, in order to interject ϕ into one of these equations, we need to raise (1) to the eighth power, getting

$$a^8 = (\sqrt{11} + \sqrt{5}\,)^8\, u_+ + (\sqrt{11} - \sqrt{5}\,)^8\, u_- \,, \tag{4}$$

which, in the standard basis is

$$
\begin{aligned}
a^8 &= \tfrac{1}{2}[(\sqrt{11}+\sqrt{5}\,)^8 + (\sqrt{11}-\sqrt{5}\,)^8] + \tfrac{1}{2}[(\sqrt{11}+\sqrt{5}\,)^8 - (\sqrt{11}-\sqrt{5}\,)^8]\,u \\
&= \tfrac{1}{2}\phi + \tfrac{1}{2}Bu\,, \qquad (5)
\end{aligned}
$$

where

$$B \equiv (\sqrt{11}+\sqrt{5}\,)^8 - (\sqrt{11}-\sqrt{5}\,)^8\,. \qquad (6)$$

However, we won't need B.

Now, we have (3), which is a pure unipode (i.e., has only numbers in it) and we have a unipode in (5) that contains our unknown ϕ. We just need to find a 'simple' relation between them. Actually, we do! Let

$$\langle\, a^8 \,\rangle_0 = \langle\, a^8 \,\rangle_0\,, \qquad (7)$$

where we'll substitute into the LHS the a^8 from (5) and into the RHS the a^8 from (3). Thus,

$$\langle\, \tfrac{1}{2}\phi + \tfrac{1}{2}Bu \,\rangle_0 = \langle\, \big(\sqrt{11}+\sqrt{5}\,u\big)^8 \,\rangle_0\,. \qquad (8)$$

The LHS simplifies to just $\frac{1}{2}\phi$. When we expand the RHS, we're going to need the binomial coefficients provided by Pascal's Triangle, which are

$$\mathbf{1},\quad 8\quad \mathbf{28},\quad 56,\quad \mathbf{70},\quad 56,\quad \mathbf{28},\quad 8,\quad \mathbf{1}\,. \qquad (9)$$

But since the selectors have been specially setup to choose only the scalar components (those where u will be to an even power), we only need the bold entries. Thus,

$$
\begin{aligned}
\tfrac{1}{2}\phi &= (1)(\sqrt{11})^8 + (28)(\sqrt{11})^6(\sqrt{5})^2 + (70)(\sqrt{11})^4(\sqrt{5})^4 + (28)(\sqrt{11})^2(\sqrt{5})^6 + (\sqrt{5})^8 \\
&= 11^4 + (28)(11)^3(5) + (70)(11)^2(5)^2 + (28)(11)(5)^3 + (5)^4 \\
&= 14614 + 186340 + 211750 + 38500 + 625 \\
&= 451856\,. \qquad (10)
\end{aligned}
$$

Therefore,

$$\phi = 903,712\,. \qquad (11)$$