Olympiad Problem 151

P. Reany

January 7, 2026

Mathematics is the art of reducing any
problem to linear algebra.
— William Stein

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Source: https://www.youtube.com/watch?v=k0BOCf7CO_g
Title: The Square Root Trap!
 Presenter: SyberMath
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1 Problem

Given the relation

$$x - \sqrt{a - x^2} = 1\,, \tag{1}$$

solve for real values of x.

2 Solution

I'll take as my first unipode,

$$\begin{aligned} b &= x u_+ + \sqrt{a - x^2}\, u_- & \text{(2a)} \\ &= \tfrac{1}{2}[x + \sqrt{a - x^2}] + \tfrac{1}{2}[x - \sqrt{a - x^2}]u & \text{(2b)} \\ &= \tfrac{1}{2}B + \tfrac{1}{2}u\,. & \text{(2c)} \end{aligned}$$

where we used (1) and where

$$B \equiv x + \sqrt{a - x^2}\,. \tag{3}$$

On squaring b from line (2c), we have that

$$b^2 \equiv \frac{B^2}{4} + \frac{1}{4} + \frac{B}{2}u\,. \tag{4}$$

So, let's recap. We got this last form for b^2 by first converting to standard form and then squaring b. Next, we'll convert b from line (2a) to standard form and then square that. After that, we'll equate, respectively, the complex and uniplex parts of both forms of b^2.

$$b^2 = x^2 u_+ + (a - x^2) u_- \tag{5a}$$

$$= \frac{a}{2} + [x^2 - \frac{a}{2}] u \tag{5b}$$

On equating the uniplex part of (5b) with the uniplex part of (4), we have that

$$x^2 - \frac{a}{2} = \frac{B}{2}\,. \tag{6}$$

Now, if we equate the complex part of (5b) with the complex part of (4), we have that

$$\frac{a}{2} = \frac{B^2}{4} + \frac{1}{4}\,. \tag{7}$$

If we add Eqs. (6) and (7), we get

$$x^2 = \frac{B}{2} + \frac{B^2}{4} + \frac{1}{4} = \left(\frac{B}{2} + \frac{1}{2}\right)^2 . \tag{8}$$

Form this we have that

$$x = \pm\left(\frac{B}{2} + \frac{1}{2}\right) . \tag{9}$$

But from (7), we can solve for B as a function of a:

$$B = \pm\sqrt{2a-1}\,. \tag{10}$$

Hence, tentatively,

$$x = \pm\left(\frac{\pm\sqrt{2a-1}}{2} + \frac{1}{2}\right) = \pm\frac{1}{2}\left(1 \pm \sqrt{2a-1}\right) . \tag{11}$$

However, x cannot have four roots. But since we're looking for real roots, the $\sqrt{a - x^2} > 0$ in (1), forcing x to be positive. Therefore,

$$x = \frac{1}{2}\left(1 \pm \sqrt{2a-1}\right) . \tag{12}$$