Olympiad Problem 35

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Abstract

Here we use the unipodal algebra to assist in solving the problem, which is given to us on YouTube. Although I'm referring to the series under the name 'olympiad', the problems are from diverse sources as olympiads, entrance exams, SATs, and the like.

> Linear Algebra is about the relation between
> the columns and the rows.
> — Gilbert Strang

The YouTube video is found at:

`Source: https://www.youtube.com/watch?v=4FPdSXaydHA`
`Title: A very tricky Oxford University Exponential Question`
`Presenter: Super Academy`

1 The Problem

Given the relation

$$(\sqrt{2}+1)^x + (\sqrt{2}-1)^x = 34\,, \tag{1}$$

find the values of x.

2 The Prerequisites: The unipodal algebra

This algebra is formed as the extension of the complex numbers by the number u, where $u^2 = 1$, and u commutes with the complex numbers. The number u is said to be 'unipotent'. The set of numbers constructed this way are the unipodal numbers, a particular such number is called a unipode. The main conjugation operator on unipode a is the unegation operator, written a^-. It does not affect complex numbers, but it sends every u to its negative. Hence, if $a = x + yu$, where x, y are complex numbers, then $a^- = x - yu$. Unegation distributes over addition and multiplication.

The following are some properties that will come in handy:

$$u^2 = 1\,, \tag{2a}$$
$$u_\pm \equiv \tfrac{1}{2}(1 \pm u)\,, \tag{2b}$$
$$u_\pm^2 = u_\pm\,, \tag{2c}$$
$$u = u_+ - u_-\,, \tag{2d}$$
$$u_+u_- = 0\,, \tag{2e}$$
$$u_+ + u_- = 1\,, \tag{2f}$$
$$uu_+ = u_+\,, \tag{2g}$$
$$uu_- = -u_-\,, \tag{2h}$$
$$(u_\pm)^- = u_\mp\,. \tag{2i}$$

You should prove (2c) – (2i). By the way, these two special unipodes $u_\pm$ square to themselves. Such numbers in a ring are referred to as *idempotents.* In the unipodal numbers they have no inverses. The fact that the unipodal number system is not a field is of little concern to me. In fact, most unipodes have inverses, so long as they are not multiples of one of the idempotents. If one needs field elements, the scalars of the unipodal numbers comprise the field of complex numbers.

Two often-used results are, for complex numbers w, z (which are used to convert unipodes between the bases $\{1, u\}$ and $\{u_+, u_-\}$):

$$w + zu = (w + z)u_+ + (w - z)u_-\,, \tag{3a}$$
$$wu_+ + zu_- = \tfrac{1}{2}(w + z) + \tfrac{1}{2}(w - z)u\,. \tag{3b}$$

Next, we learn how to take the 'norm' of a unipode. Let w be a unipode in standard basis, given by

$$w = a + bu\,, \tag{4}$$

where a, b are complex numbers. The 'norm' of w is given as[1]

$$ww^- = (a + bu)(a - bu) = a^2 - b^2\,. \tag{5}$$

Now, let y be a unipode in idempotent basis, given as

$$y = Au_+ + Bu_-\,, \tag{6}$$

where A, B are complex numbers. The 'norm' of y is given as

$$\begin{aligned} yy^- &= (Au_+ + Bu_-)(Au_- + Bu_+) = ABu_+ + ABu_- \\ &= AB(u_+ + u_-) = AB\,. \end{aligned} \tag{7}$$

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[1] Calling ww^- a 'norm' is rather imprecise. In accordance with terminology used by G. Sobczyk, I will call $\left| ww^- \right|^{1/2}$ the (unipodal) modulus, and ww^- the (unipodal) di-modulus of w. See the Appendix.

The unipodal algebra has two copies of the complex numbers, one for each component. In any true unipodal equation, the corresponding coefficients across the equal sign are equal to each other. This is similar to equating real and imaginary components across the equal sign in the complex algebra.

When I first used the unipodal algebra to solve polynomial equations (c. 1984-5), I used the Clifford 1 algebra over the complex numbers. The '1' means one unit vector u. So, a Clifford 1 number c can be represented as

$$c = a + bu\,, \tag{8}$$

where a, b are complex numbers. Of course, u being a unit vector, then

$$u^2 = 1\,. \tag{9}$$

Now, the standard basis for this space is $\{1, u\}$ and the scalars are the complex numbers. To extract the 'scalar part' of (8), we use the selection operator $\langle\,\cdot\,\rangle$, as follows:

$$\langle\, c\,\rangle = \langle\, a + bu\,\rangle = a\,, \tag{10}$$

One can also subscript the selector with a zero for the scalar part:

$$\langle\, c\,\rangle_0 = \langle\, a + bu\,\rangle_0 = a\,, \tag{11}$$

and with a '1' for the vector part:

$$\langle\, c\,\rangle_1 = \langle\, a + bu\,\rangle_1 = bu\,. \tag{12}$$

When I adopted the name 'unipodal algebra' from a paper I cowrote with two other authors, I found a need to adopt new terminology for naming the scalar and vector parts. Just as complex numbers are composed of a real number times the unit '1' and another real number times the unit imaginary i, the unipodal numbers are composed of a complex number times the unit '1' and another complex number times the unipotent number u. The part of the unipode that does not contain the unipotent factor is called the 'complex part' of the unipode. The part that does contain the unipotent element factor is called the **uniplex part** of the unipode.

Now, before you complain that calling the scalar part of a unipode the 'complex part' is nonsense, I point out that in complex analysis, the nonimaginary part is referred to as the 'real part'. Lastly, when I say the 'uniplex part' in this series of papers, I refer only to the coefficient of the nonscalar part, which is complex only. Thus the uniplex part of unipode $c = a + bu$ is just b. Another way to think of the uniplex part of c is to take the scalar (or complex) part of cu.

$$\langle\, c\,\rangle_1 = \langle\, cu\,\rangle = \langle\, au + b\,\rangle = b\,. \tag{13}$$

Thus, one must be careful when I report I'm taking the uniplex part of a unipode (across all the papers I've written over the years), because at times it may contain that factor of u and at other times not. But like I said: In this series it will always mean only the scalar factor of the unipotent element.

Much of the algebraic power of the unipodal algebra comes from 1) it being able to switch the presentation of a unipode between the standard basis and the idempotent basis, the latter basis being well suited for taking powers and roots.

3 The Solution

First, we define the auxiliary parameter k:

$$k \equiv \tfrac{1}{2}[\,(\sqrt{2}+1)^x - (\sqrt{2}-1)^x\,]\,. \tag{14}$$

Let our first unipode be

$$\begin{aligned} a &= (\sqrt{2}+1)\,u_+ + (\sqrt{2}-1)\,u_- && \text{(15a)}\\ &= \sqrt{2} + u\,. && \text{(15b)} \end{aligned}$$

Now for the di-modulus of a:

$$aa^- = (\sqrt{2})^2 - 1^2 = 1\,. \tag{16}$$

Now it's time to reach the ceiling:

$$\begin{aligned} a^x &= (\sqrt{2}+1)^x\,u_+ + (\sqrt{2}-1)^x\,u_- && \text{(17a)}\\ &= \tfrac{1}{2}[\,(\sqrt{2}+1)^x + (\sqrt{2}-1)^x\,] + \tfrac{1}{2}[\,(\sqrt{2}+1)^x - (\sqrt{2}-1)^x\,]\,u && \text{(17b)}\\ &= 17 + ku\,, && \text{(17c)} \end{aligned}$$

where we used (1) and (14).

For the di-modulus of a^x,we have:

$$a^x a^{x-} = 17^2 - k^2 = (aa^-)^x = 1\,. \tag{18}$$

From this we get for k:

$$k = \pm 12\sqrt{2}\,. \tag{19}$$

Putting (1) and (14) together:

$$\begin{aligned} (\sqrt{2}+1)^x + (\sqrt{2}+1)^x &= 34\,, && \text{(20a)}\\ (\sqrt{2}+1)^x - (\sqrt{2}+1)^x &= \pm 12\sqrt{2}\,. && \text{(20b)} \end{aligned}$$

On adding them, and then subtracting them, we get (with simplification)

$$\begin{aligned} (\sqrt{2}+1)^x &= 17 \pm 12\sqrt{2}\,, && \text{(21a)}\\ (\sqrt{2}-1)^x &= 17 \mp 12\sqrt{2}\,. && \text{(21b)} \end{aligned}$$

How shall we proceed? If we multiply these together, we'll just end up with $1 = 1$. So, let's divide them (and get ready for a lot of computation).

$$\frac{(\sqrt{2}+1)^x}{(\sqrt{2}-1)^x} = \frac{17 \pm 12\sqrt{2}}{17 \mp 12\sqrt{2}}\,. \tag{22}$$

I'll just do one of these computations for proof of concept.

$$\frac{(\sqrt{2}+1)^x}{(\sqrt{2}-1)^x} = \frac{17+12\sqrt{2}}{17-12\sqrt{2}}\,. \tag{23}$$

Then

$$\frac{(\sqrt{2}+1)^x}{(\sqrt{2}-1)^x}\frac{(\sqrt{2}+1)^x}{(\sqrt{2}+1)^x} = \frac{17+12\sqrt{2}}{17-12\sqrt{2}}\frac{17+12\sqrt{2}}{17+12\sqrt{2}}\,. \tag{24}$$

So,

$$(\sqrt{2}+1)^{2x} = (17+12\sqrt{2})^2\,. \tag{25}$$

Thus,

$$(\sqrt{2}+1)^x = 17+12\sqrt{2}\,. \tag{26}$$

Given that this is one of those fancy 'olympiad' style problems, it's time to forego using general methods (such as taking logarithms across that last equation) and look for a simple solution. In that vein, let's just assume that x is a positive integer. So, we can start at 1 and keep increasing the value until, hopefully, we get a solution.

Well, $x = 1, 2, 3$ won't work, but what about $x = 4$? Actually, it works. Remember, though, that the result a only the 'proof of concept' case.

4 Conclusion

Let's take a moment to briefly take stock of the unipodal techniques we've used so far in this series that have been useful (and add in one or two that might be useful in the future):

1) Forming the 'first unipode' wisely.

2) Taking roots or powers, especially on unipodes in the idempotent basis.

3) 'Flipping' between bases.

4) Extracting the complex and/or uniplex parts across an equation.

5) Taking the 'magnitude square' of a unipode. For example, if $X = x_0 + x_1 u$, $XX^- = x_0^2 - x_1^2$, which is, of course, just a complex number. If two unipodes are equal, their square magnitudes are equal, and you are free to calculate their square magnitudes from either basis.

6) Comparing square magnitudes this way: $X^n(X^-)^n = (XX^-)^n$.

6) If A and B are equal unipodes in standard form, then $\dfrac{a_0}{a_1} = \dfrac{b_0}{b_1}$, but if they are in idempotent form, then $\dfrac{a_+}{a_-} = \dfrac{b_+}{b_-}$.

Furthermore, we can add to these tricks all the techniques of real and complex number and ring theory.

5 Appendix

Here I want to present a bit more theory on the unipodal algebra. I've found the need to do this because the broader the space of algebra problems I try to solve with the unipodal algebra, the broader the unipodal theory I find I need to draw upon. And if I have to, the reader has to as well.

Note: I will be making references to Garret Sobczyk's book *New Foundations in Mathematics, The Geometric Concept of Number* [1], particularly in the sections he has on the unipodal and hyperbolic numbers.

Let's begin with the algebra of the **hyperbolic** extension of the real numbers. We start with the real numbers $\mathbb{R}$ and extend them by the unipotent element u. This is denoted by $\mathbb{R}[u]$. Thus, a typical **hyperbolic number** h in standard form could be

$$h = x + yu\,, \tag{27}$$

where x, y are real numbers. Flipping this to idempotent form, we get

$$h = h_+ u_+ + h_- u_- \,. \tag{28}$$

For considerations due to symmetric 2×2 matrices, Sobczyk calls the process of going from (27) to (28) the *spectral decomposition* of (27) ([1], p. 33). I suppose we could call this the 'spectral basis'. However, we will stick with calling it the 'idempotent basis'.

Let w be a general unipodal number for starters. Now, if w is neither zero nor a multiple of one of the idempotents, then it will have an inverse. The easiest way to find the inverse of w is to cast it first into the idempotent basis, like this:

$$w = w_+ u_+ + w_- u_- \,. \tag{29}$$

Then its inverse is

$$w^{-1} = w_+^{-1} u_+ + w_-^{-1} u_- = \frac{1}{w_+} u_+ + \frac{1}{w_-} u_- \,. \tag{30}$$

Clearly, this inverse exists because neither w_+ nor w_- is zero, which we know to be the case because if either of them were zero, then w in (29) would reduce to being a multiple of one of the idempotents, which we have disallowed.

Next, comes the important issue of defining some sort of magnitude on the unipodal numbers, starting with the hyperbolic numbers. For hyperbolic number h, we can define the **hyperbolic modulus** by ([1], p. 25):

$$|\, h \,|_h \equiv \sqrt{|\, hh^- \,|}\,, \tag{31}$$

where, of course, hh^- is a real number.

Now, we can define something similar for unipodal numbers, such as for unipodal number w, we can define the **unipodal modulus** by:

$$|\, w \,|_u \equiv \sqrt{|\, ww^- \,|}\,, \tag{32}$$

where, of course, ww^- is a complex number, but $|\, w \,|_u$ is a real number.

Now, I know why the real numbers play such an important role in the hyperbolic numbers, that being its close association with the hyperbolic plane and Lorentzian geometry. But it's been my experience in using the unipodal algebra to solve problems, that magnitudes of them represented by real numbers have not played much, if any, role (at least so far). Therefore, I propose to define a more useful notion of modulus for what I'm doing.

For unipodal number w, we can define the **unipodal di-modulus** by:

$$\mathrm{mod}\,(w) = ww^-\,, \tag{33}$$

where, of course, ww^- is a complex number. The meaning of 'di-modulus' is this: The 'di' part refers to two aspects of the complex number ww^-, that being its magnitude and complex phase. And by not introducing squareroots, we refrain from burdening the algebra with unnecessary algebraic complications.

Theorem: If w is a unipode such that

$$ww^- = 1\,, \tag{34}$$

then

$$w^{-1} = w^-\,. \tag{35}$$

Proof:

Clearly, w is not zero, nor is it a multiple of an idempotent. Let's prove this by contradiction. Assume that

$$w = \alpha u_+\,, \tag{36}$$

where α is a complex number. Then

$$ww^- = (\alpha u_+)(\alpha u_-) = \alpha^2 u_+ u_- = 0\,. \tag{37}$$

But ww^- cannot be both unity and zero at the same time, hence, a contradiction. Therefore w is not a multiple of u_+; and by a similar argument, it is not a multiple of u_-.

Thus we know that w^{-1} exists. Therefore, multiplying across (34), we have that

$$w^{-1}(ww^-) = (w^{-1}w)w^- = w^{-1}\,. \tag{38}$$

And thus,

$$w^- = w^{-1}\,. \tag{39}$$

Lemma: If a, b are unipodes, then

$$a^{-}b^{-} = (ab)^{-}\,. \tag{40}$$

Proof: Hint: Set $a = a_{+}u_{+} + a_{-}u_{-}$ and $b = b_{+}u_{+} + b_{-}u_{-}$ and work it out.

Theorem: If w is a unipode then for positive integer n

$$(w^{-})^{n} = (w^{n})^{-}\,. \tag{41}$$

Proof: (By induction) For $n = 1$ there's nothing to show.

Multiply (41) through by w^{-}:

$$w^{-}(w^{-})^{n} = w^{-}(w^{n})^{-}\,. \tag{42}$$

The LHS becomes $(w^{-})^{n+1}$ by ordinary product-counting rules. The RHS becomes $(w^{n+1})^{-}$ by the previous lemma. Therefore,

$$(w^{-})^{n+1} = (w^{n+1})^{-}\,. \tag{43}$$

So, by assuming that the rule is true for case n, we were able to show that the rule also works for case $n + 1$. And we're done.

References

[1] G. Sobczyk, *New Foundations in Mathematics, The Geometric Concept of Number*, Birkhauser/Springer, New York, 2013.