Olympiad Problem 46

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Abstract

Here we use the unipodal algebra to assist in solving the problem, which is given to us on YouTube. Although I'm referring to the series under the name 'olympiad', the problems are from diverse sources as olympiads, entrance exams, SATs, and the like.

You will never plough a field if you only turn
it over in your mind.
— Irish Proverb

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=xSITkGFst_A
Title: South Africa Math Olympiad Question
Presenter: LKLogic
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1 The Problem

Given the relations

$$x^2 - y^2 = 24\,, \tag{1}$$

and

$$xy = 35\,, \tag{2}$$

find the solutions.[1]

Note: This is a repeat of Olympiad Problem 4, but done in a different fashion.

Note: WolframAlpha.com gives the solutions as $(7, 5), (-7, -5), (5i, -7i), (-5i, 7i)$.

[1]The problem, as given, only requested the sums of $x + y$, but I will get the four roots.

2 The Prerequisites: The unipodal algebra

This algebra is formed as the extension of the complex numbers by the number u, where $u^2 = 1$, and u commutes with the complex numbers. The number u is said to be 'unipotent'. The set of numbers constructed this way are the unipodal numbers, a particular such number is called a unipode. The main conjugation operator on unipode a is the unegation operator, written a^-. It does not affect complex numbers, but it sends every u to its negative. Hence, if $a = x + yu$, where x, y are complex numbers, then $a^- = x - yu$. Unegation distributes over addition and multiplication.

The following are some properties that will come in handy:

$$\begin{aligned} u^2 &= 1\,, && \text{(3a)}\\ u_\pm &\equiv \tfrac{1}{2}(1 \pm u)\,, && \text{(3b)}\\ u_\pm^2 &= u_\pm\,, && \text{(3c)}\\ u &= u_+ - u_-\,, && \text{(3d)}\\ u_+u_- &= 0\,, && \text{(3e)}\\ u_+ + u_- &= 1\,, && \text{(3f)}\\ uu_+ &= u_+\,, && \text{(3g)}\\ uu_- &= -u_-\,, && \text{(3h)}\\ (u_\pm)^- &= u_\mp\,. && \text{(3i)} \end{aligned}$$

You should prove (3c) – (3i). By the way, these two special unipodes $u_\pm$ square to themselves. Such numbers in a ring are referred to as *idempotents*. In the unipodal numbers they have no inverses. The fact that the unipodal number system is not a field is of little concern to me. In fact, most unipodes have inverses, so long as they are not multiples of one of the idempotents. If one needs field elements, the scalars of the unipodal numbers comprise the field of complex numbers.

Two often-used results are, for complex numbers w, z (which are used to convert unipodes between the bases $\{1, u\}$ and $\{u_+, u_-\}$):

$$w + zu = (w + z)u_+ + (w - z)u_-\,, \qquad \text{(4a)}$$

$$wu_+ + zu_- = \tfrac{1}{2}(w + z) + \tfrac{1}{2}(w - z)u\,. \qquad \text{(4b)}$$

Much of the algebraic power of the unipodal algebra comes from 1) it being able to switch the presentation of a unipode between the standard basis and the idempotent basis, the latter basis being well suited for taking powers and roots. It reminds me of when I was a kid, and other kids would fold a piece of paper in such a way that they could, with two fingers of each hand, open and close the folded paper in two different ways. The practice of this folding is called origami. (Some call the result of that folding the 'Fortune Teller' fold.) But I think of this construction as an analogy: The paper represents a unipode: Open it one way to see the number in the standard basis, and open it the other way to see it in the idempotent basis.

3 The Solution

Let's take as our 'first unipode':

$$a \equiv x + iyu\,. \tag{5}$$

Now we square it:

$$a^2 = x^2 - y^2 + 2ixyu\,. \tag{6}$$

Using the given information, we get

$$a^2 = 24 + 70iu = 2(12 + 35iu)\,, \tag{7}$$

which is a pure unipode, and thus it should be easy to take its square root. To facilitate taking its square root, we flip the bases:

$$a^2 = 2[(12 + 35i)\,u_+ + (12 - 35i)\,u_-\,]\,. \tag{8}$$

Formally, the square root looks like this:

$$a = \sqrt{2}\,[(12 + 35i)^{1/2}\,u_+ \pm (12 - 35i)^{1/2}\,u_-\,]\,. \tag{9}$$

Of course, when we take square roots, we get plus and minus signs, and the trick is to weave them together correctly.[2]

I used Mathematica to find the square roots, and thus

$$a = \sqrt{2}\,\Big[\,\left(\frac{5i \pm 7}{\sqrt{2}}\right) u_+ \pm \left(\frac{-5i \pm 7}{\sqrt{2}}\right) u_-\,\Big]\,. \tag{10}$$

After I cancel the square root of 2, I intend to flip the basis back to the standard basis, so that I can do a direct equating of x and iy in (5).

$$a = \tfrac{1}{2}\,[\,(5i \pm 7) \pm (-5i \pm 7)\,] \pm \tfrac{1}{2}\,[\,(5i \pm 7) \mp (-5i \pm 7)\,]\,u\,. \tag{11}$$

First, we'll look at the x values:[3]

$$x = \tfrac{1}{2}\,[\,(5i \pm 7) \pm (-5i \pm 7)\,]\,, \tag{12a}$$

$$= \begin{cases} \tfrac{1}{2}\,[\,(5i + 7) \pm (-5i + 7) = 7 \text{ (on `+')}, 5i \text{ (on `-')} \\ \tfrac{1}{2}\,[\,(5i - 7) \pm (-5i - 7) = -7 \text{ (on `+')}, 5i \text{ (on `-')} \end{cases} \tag{12b}$$

[2]The $\pm$ sign between the components is because the square roots of the components are taken independently of each other.

[3]Warning: When we extract square roots in the unipodal algebra, we get a lot of extraneous roots. Our first attempt at solving for roots will not give us all the roots, but will contain extraneous roots. The fix is pretty easy: we adjust the signs in front of the terms until we get solutions to Eq. (2). And if we're really energetic, we can do the same for (1).

Next, we'll look at the y values:

$$iy = \tfrac{1}{2}\left[\,(5i \pm 7) \mp (-5i \pm 7)\,\right], \tag{13a}$$

$$= \begin{cases} \tfrac{1}{2}\left[\,(5i + 7) \mp (-5i + 7) = 7 \text{ (on `+')}, 5i \text{ (on `-')} \\ \tfrac{1}{2}\left[\,(5i - 7) \mp (-5i - 7) = -7 \text{ (on `+')}, 5i \text{ (on `-')} \end{cases} \tag{13b}$$

The correspondences are

$$x = 7 \qquad -7 \qquad x \qquad = 5i, \qquad 5i \tag{14a}$$

$$iy = 5i, \qquad 5i \qquad iy \qquad = 7, \qquad -7 \tag{14b}$$

$$y = 5, \qquad 5 \qquad y \qquad = -7i, \qquad 7i \tag{14c}$$

So, now we pair the x's and y's up in columns, yielding

$$(7,5),\ (-7,5),\ (5i,-7i),\ (5i,7i)\,. \tag{15}$$

And now we deal with the extraneous root problem: First, clearly $(7,5)$ and $(-7,-5)$ satisfy (2), but $(-7,5)$ does not. Also, $(5i,-7i)$ satisfies (2), but $(5i,7i)$ does not. But $(-5i,7i)$ does. And that's four root pairs.

$$(7,5),\ (-7,-5),\ (5i,-7i),\ (-5i,7i)\,. \tag{16}$$

So, how do we justify adding in these minus signs? Well, way back in (10), I really should have put in another $\pm$ sign in front of the u_+ term. I didn't because it's too confusing. It's not an elegant fix, but dealing with the extraneous roots is not fun in conventional methods either.

My personal opinion is that these intensive calculations of square roots of complex numbers should be done by computers that understand the unipodal algebra.