Olympiad Problem 5

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January 27, 2025

Abstract

Here we use the unipodal algebra to assist in solving the problem, which is given to us on YouTube. Although I'm referring to the series under the name 'olympiad', the problems are from diverse sources as olympiads, entrance exams, SATs, and the like.

Physical concepts are free creations of the human mind, and
are not, however it may seem, uniquely determined
by the external world.
— Albert Einstein

The YouTube video is found at:

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Source: https://www.youtube.com/shorts/5Rfa6lDU3XY
Title: A Nice Olympiad Algebra Problem
 Presenter:  Numbers & Numbers
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1 The Problem

Given the relation

$$a + \frac{1}{a} = 6\,, \tag{1}$$

find

$$a - \frac{1}{a}\,, \tag{2}$$

over the complex numbers.

2 The Prerequisites: The unipodal algebra

This algebra is formed as the extension of the complex numbers by the number u, where $u^2 = 1$, and u commutes with the complex numbers. The number u is said to be 'unipotent'. The set of numbers constructed this way are the unipodal numbers, a particular such number is called a unipode. The main conjugation operator on unipode a is the unegation operator, written a^-. It does not affect complex numbers, but it sends every u to its negative. Hence, if $a = x + yu$, where x, y are complex numbers, then $a^- = x - yu$. Unegation distributes over addition and multiplication.

The following are some properties that will come in handy:

$$u^2 = 1\,, \tag{3a}$$
$$u_\pm \equiv \tfrac{1}{2}(1 \pm u)\,, \tag{3b}$$
$$u_\pm^2 = u_\pm\,, \tag{3c}$$
$$u = u_+ - u_-\,, \tag{3d}$$
$$u_+ u_- = 0\,, \tag{3e}$$
$$u_+ + u_- = 1\,, \tag{3f}$$
$$uu_+ = u_+\,, \tag{3g}$$
$$uu_- = -u_-\,, \tag{3h}$$
$$(u_\pm)^- = u_\mp\,. \tag{3i}$$

You should prove (3c) – (3i). By the way, these two special unipodes $u_\pm$ square to themselves. Such numbers in a ring are referred to as *idempotents*. In the unipodal numbers they have no inverses. The fact that the unipodal number system is not a field is of little concern to me. In fact, most unipodes have inverses, so long as they are not multiples of one of the idempotents. If one needs field elements, the scalars of the unipodal numbers comprise the field of complex numbers.

Two often-used results are, for complex numbers w, z (which are used to convert unipodes between the bases $\{1, u\}$ and $\{u_+, u_-\}$):

$$w + zu = (w + z)u_+ + (w - z)u_-\,, \tag{4a}$$
$$wu_+ + zu_- = \tfrac{1}{2}(w + z) + \tfrac{1}{2}(w - z)u\,. \tag{4b}$$

For unipodal number w, we can define the **unipodal di-modulus** by:

$$\operatorname{mod}(w) = ww^-\,, \tag{5}$$

where, of course, ww^- is a complex number. The meaning of 'di-modulus' is this: The 'di' part refers to two aspects of the complex number ww^-, that being its magnitude and complex phase. And by not introducing squareroots, we refrain from burdening the algebra with unnecessary algebraic complications.

If

$$w = au_+ + bu_- = c + du\,, \tag{6}$$

then

$$ww^- = ab = c^2 - d^2\,, \tag{7}$$

which is often a quick way to get a useful result. This is especially true in cases when $ab = 1$.

Much of the algebraic power of the unipodal algebra comes from 1) it being able to switch the presentation of a unipode between the standard basis and the idempotent basis, the latter basis being well suited for taking powers and roots. It reminds me of when I was a kid, and other kids would fold a piece of paper in such a way that they could, with two fingers of each hand, open and close the folded paper in two different ways. The practice of this folding is called origami. (Some call the result of that folding the 'Fortune Teller' fold.) But I think of this construction as an analogy: The paper represents a unipode: Open it one way to see the number in the standard basis, and open it the other way to see it in the idempotent basis.

3 The First Solution

Let's give our unknown quantity a name:

$$a - \frac{1}{a} \equiv k\,. \tag{8}$$

We define the unipode X by

$$X \equiv a + \frac{1}{a}u\,. \tag{9}$$

When we convert a to the idempotent basis to pick-up 6 and k, we get

$$X = \big(a + \frac{1}{a}\big)\,u_+ + \big(a - \frac{1}{a}\big)\,u_- = 6u_+ + k u_-\,. \tag{10}$$

Next, we convert back to standard basis:

$$X = \tfrac{1}{2}(6+k) + \tfrac{1}{2}(6-k)u\,. \tag{11}$$

Now, if we take the product of the components of (9) and set them equal to the product of the components of (11), we get

$$1 = \tfrac{1}{4}(6+k)(6-k)\,. \tag{12}$$

Solving this for k, we have that

$$k = a - \frac{1}{a} = \pm 4\sqrt{2}\,. \tag{13}$$

4 The Second Solution

As before, let's give our unknown quantity a name:

$$a - \frac{1}{a} \equiv k\,. \tag{14}$$

We define the unipode Y by

$$Y \equiv a u_+ + \frac{1}{a} u_- \,. \tag{15}$$

On calculating the di-modulus of Y, we get

$$Y Y^- = a \frac{1}{a} = 1 \,. \tag{16}$$

Next, we go back to (15) and flip the basis:

$$Y = \frac{1}{2}\big(a + \frac{1}{a}\big) + \frac{1}{2}\big(a - \frac{1}{a}\big) u = 3 + \frac{k}{2} u \,. \tag{17}$$

Calculating the di-modulus of Y from this last equation and then setting it equal to unity, we get

$$9 - \frac{k^2}{4} = 1 \,. \tag{18}$$

Solving for k, we get, as before

$$k = a - \frac{1}{a} = \pm 4\sqrt{2} \,. \tag{19}$$