Olympiad Problem 7

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Abstract

Here we use the unipodal algebra to assist in solving the problem, which is given to us on YouTube. Although I'm referring to the series under the name 'olympiad', the problems are from diverse sources as olympiads, entrance exams, SATs, and the like.

In the middle of difficulty lies opportunity.
— John Archibald Wheeler

The YouTube video is found at:

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Source:  https://www.youtube.com/watch?v=fX-FrWzh-Vs
 Titled: Can you Solve Harvard University System of Equations ?
  Presenter: Super Academy
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1 The Problem

Given the relations

$$10^2 x + 10y^2 = 29xy\,, \tag{1a}$$

$$x^2 - y^2 = 21\,, \tag{1b}$$

find

$$x + y\,, \tag{2}$$

where x, y are positive real numbers.

2 The Prerequisites: The unipodal algebra

This algebra is formed as the extension of the complex numbers by the number u, where $u^2 = 1$, and u commutes with the complex numbers. The number u is said to be 'unipotent'. The set of numbers constructed this way are the unipodal numbers, a particular such number is called a unipode. The main conjugation operator on unipode a is the unegation operator, written a^-. It does not affect complex numbers, but it sends every u to its negative. Hence, if $a = x + yu$, where x, y are complex numbers, then $a^- = x - yu$. Unegation distributes over addition and multiplication.

The following are some properties that will come in handy:

$$u^2 = 1\,, \tag{3a}$$

$$u_\pm \equiv \tfrac{1}{2}(1 \pm u)\,, \tag{3b}$$

$$u_\pm^2 = u_\pm\,, \tag{3c}$$

$$u = u_+ - u_-\,, \tag{3d}$$

$$u_+ u_- = 0\,, \tag{3e}$$

$$u_+ + u_- = 1\,, \tag{3f}$$

$$u u_+ = u_+\,, \tag{3g}$$

$$u u_- = -u_-\,, \tag{3h}$$

$$(u_\pm)^- = u_\mp\,. \tag{3i}$$

You should prove (3c) – (3i). By the way, these two special unipodes $u_\pm$ square to themselves. Such numbers in a ring are referred to as *idempotents*. In the unipodal numbers they have no inverses. The fact that the unipodal number system is not a field is of little concern to me. In fact, most unipodes have inverses, so long as they are not multiples of one of the idempotents. If one needs field elements, the scalars of the unipodal numbers comprise the field of complex numbers.

Two often-used results are, for complex numbers w, z (which are used to convert unipodes between the bases $\{1, u\}$ and $\{u_+, u_-\}$):

$$w + zu = (w + z)u_+ + (w - z)u_-\,, \tag{4a}$$

$$wu_+ + zu_- = \tfrac{1}{2}(w + z) + \tfrac{1}{2}(w - z)u\,. \tag{4b}$$

Much of the algebraic power of the unipodal algebra comes from 1) it being able to switch the presentation of a unipode between the standard basis and the idempotent basis, the latter basis being well suited for taking powers and roots. It reminds me of when I was a kid, and other kids would fold a piece of paper in such a way that they could, with two fingers of each hand, open and close the folded paper in two different ways. The practice of this folding is called origami. (Some call the result of that folding the 'Fortune Teller' fold.) But I think of this construction as an analogy: The paper represents a unipode: Open it one way to see the number in the standard basis, and open it the other way to see it in the idempotent basis.

3 The Solution

Let our 'first unipode' be defined as follows:

$$b \equiv x + yu = (x + y)u_+ + (x - y)u_-\,. \tag{5}$$

$$b \equiv x + yu = (x+y)u_+ + (x-y)u_- \tag{6a}$$

$$b^2 = x^2 + y^2 + 2xyu = xy\Big(\frac{29}{10} + 2u\Big) \quad \text{(using (1a))}\,, \tag{6b}$$

$$b^2 = (x+y)^2 u_+ + (x-y)^2 u_- \,. \tag{6c}$$

We need to find some way to conveniently work-in the given constraints, first, because they are required, and, second, because they help us to simplify the equations.[1] To that end, let's recast b^2 of (6b) into idempotent form so that we can compare components from it to (6c).

$$b^2 = xy\Big[\big(\frac{29}{10} + 2\big)u_+ + \big(\frac{29}{10} - 2\big)u_-\Big]\,. \tag{7}$$

Now let's compare the product of the components of (6c) with that of (7):

$$(x+y)^2(x-y)^2 = (x^2 - y^2)^2 = x^2y^2\big(\frac{29}{10} + 2\big)\big(\frac{29}{10} - 2\big)\,, \tag{8}$$

which, with (1b), gives us

$$21^2 = x^2y^2\Big[\big(\frac{29}{10}\big)^2 - 4\big)\Big]\,, \tag{9}$$

and this reduces to

$$xy = 10\,. \tag{10}$$

Using this and the u_+ components of and (6c) and (7) , we get that

$$(x+y)^2 = 29 + 20 = 49\,. \tag{11}$$

Hence,

$$x + y = \pm 7\,. \tag{12}$$

[1] By 'simplify the equations' I mean roughly, in this context, 'more numbers, fewer variables'.