A unipodal integral: $\int \sinh mx \, \sinh nx \, dx$

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The purpose of this note is to prove the following identity over the complex numbers

$$\int \sinh mx \, \sinh nx \, dx = \frac{\sinh(m+n)x}{2(m+n)} - \frac{\sinh(m-n)x}{2(m-n)} \,, \tag{1}$$

where m and n are integers ($m^2 \neq n^2$) and x is complex. Our approach here will be to embed the problem in the unipodal algebra, which is the extension of the complex numbers by the unipotent number u. That is, u is a number not equal to any complex number but that has square 1 and commutes with every complex number. For our purpose we will regard u as a unit vector.

The following list of identities is obeyed by the unipodal numbers

$$u^2 = 1 \,, \tag{2a}$$
$$u_\pm \equiv \tfrac{1}{2}(1 \pm u) \,, \tag{2b}$$
$$u_+ + u_- = 1 \,, \tag{2c}$$
$$u_\pm^2 = u_\pm \,, \tag{2d}$$
$$u u_\pm = \pm u_\pm \,, \tag{2e}$$
$$u = u_+ - u_- \,, \tag{2f}$$
$$u_+ u_- = 0 \,, \tag{2g}$$
$$e^{xu} = \cosh x + u \sinh x \,, \tag{2h}$$
$$e^{-xu} = \cosh x - u \sinh x \,, \tag{2i}$$
$$\cosh x = \tfrac{1}{2}(e^{xu} + e^{-xu}) = \tfrac{1}{2}(e^{x} + e^{-x}) \,, \tag{2j}$$
$$\sinh x = \tfrac{1}{2}u(e^{xu} - e^{-xu}) = \tfrac{1}{2}(e^{x} - e^{-x}) \,, \tag{2k}$$
$$\cosh^2 x - \sinh^2 x = 1 \,, \tag{2l}$$
$$u_\pm^- = u_\mp \,, \tag{2m}$$
$$u^{-1} = u \,. \tag{2n}$$

This next fundamental theorem can be proved with elementary power series expansion of the exponential

$$e^{z_+ u_+ + z_- u_-} = e^{z_+} u_+ + e^{z_-} u_- \,, \tag{3}$$

where z_+ and z_- are the components of a unipode in idempotent basis form. This last theorem is so fundamental that there are hardly proper words to indicate its true importance. And of similar importance is the relation

$$\mathrm{Log}\,(x) \equiv u_+ \mathrm{Log}\,(x_+) + u_- \mathrm{Log}\,(x_-) \,. \tag{4}$$

Every invertible unipodal number $a + xu$ can be written in the alternative form $\lambda e^{\nu u}$, where λ and ν are complex numbers. Thus

$$\lambda e^{\nu u} = a + xu \,.$$

DEFINITION. By the *unipodal conjugate* or *unegate* of the unipode $a_0 + a_1 u$ we mean the unipode $a_0 - a_1 u$. We define the following unegation operator to accomplish getting the unegate. Let $x = a_0 + a_1 u$ be any unipode then the unegate of x is x^-. Note that $xx^- = x^- x = a_0^2 - a_1^2$ is a complex number.

Now we can get to the proof of the given identity. Let

$$\int \sinh mx \sinh nx \, dx = \mathrm{Vec} \int e^{umx} \sinh nx \, dx \,, \tag{5}$$

where "Vec" means to take the vector part, which is the coefficient of the u term.

$$\int \sinh mx \, \sinh nx \, dx = \mathrm{Vec} \int e^{umx} \left(\frac{e^{unx} - e^{-unx}}{2u} \right) dx \,, \tag{6}$$

where we used equation (2k). Now we split the integral.

$$\int \sinh mx \, \sinh nx \, dx = \mathrm{Vec} \, \frac{1}{2u} \left[\int e^{u(m+n)x} \, dx - \int (e^{u(m-n)x} \, dx \right] . \tag{7}$$

On integrating and simplifying we get

$$\int \sinh mx \, \sinh nx \, dx = \mathrm{Vec} \, \frac{1}{2} \left[\frac{e^{u(m+n)x}}{m+n} - \frac{e^{u(m-n)x}}{m-n} \right] . \tag{8}$$

And finally we use (2h) to get

$$\int \sinh mx \, \sinh nx \, dx = \frac{\sinh(m+n)x}{2(m+n)} - \frac{\sinh(m-n)x}{2(m-n)} \, . \tag{9}$$