The Variation of Air Pressure with Height Problem

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Abstract

I have recently been investigating the derivation of the variation of air pressure with height and I haven't been able to follow the presentations I've seen so far. This paper is my attempt to make a logical derivation of this formula.

1 Statement of the Problem

What is the correct derivation of the variation of air pressure (of the atmosphere) with height? Now, there are various levels of accuracy that one could strive for. The formula will contain factors for both density of air and temperature of air, and both of those will change with height. However, this attempt will treat air temperature as a constant, therefore, the result will be a crude approximation.

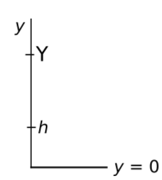

Figure 1. In this approximation to derive a formula for the pressure of air of a function of height (above sea level), we really just need the vertical axis. The $y = 0$ mark is sea level. The $y = Y$ mark is the height above sea level at which air pressure is just one Pascal. All we need for Y is its relationship to air pressure at sea level, P_0.

2 Conceptualization of the Problem

We're looking to find a formula for the air pressure as a function of height above sea level. We'll establish a coordinate system as indicated in Fig. 1. Obviously, the y-axis is the vertical axis.

The partial pressure exerted on a surface by a cylinder, resting on the surface, whose contents is a homogeneous fluid of density ρ and height of the fluid in the cylinder h, is

$$P = \rho g h \,, \tag{1}$$

where g is the local acceleration of gravity, which we will consider to be a constant for this our first approximation formula, and which ρ is the mass density, which has units [mass/vol].

Now, as we go up, pressure goes down (we know this from experience). Therefore, to first order expansion

$$P(y + dy) = P(y) + dP \,, \tag{2}$$

and if $dy > 0$, then $dP < 0$, that is, it requires a negative sign inserted at the right time. We will get to that.

To get the pressure at some height y above sea level, we will need to use an integration over dP. It so happens that the reasoning we used to get (1) will work for a differential horizontal slice of some vertical column of air. Thus,

$$dP = -\rho g dy \,, \tag{3}$$

where I needed to regress to using the variable y.

3 Using $PV = nRT$ to get rid of ρ

In Eq. (3) we have the variable ρ, which varies with height. We will substitute out ρ for the variables in the Idea Gas Equation

$$PV = nRT \,, \tag{4}$$

where R is the ideal gas constant, T is temperature in Kelvin, and n is the number of moles of the gas. To make the substitution we're looking for, we need some relation between ρ in (3) and n in (4). Let's define $\overline{m}$ as the molar mass (mass per unit mole).

$$\frac{P}{RT} = \frac{n}{V}\left[\frac{\text{moles}}{\text{vol}}\right] = \rho\left[\frac{\text{mass}}{\text{vol}}\right] \Big/ \overline{m}\left[\frac{\text{mass}}{\text{mole}}\right] . \tag{5}$$

From (3) and this last equation (solved for ρ), we have that

$$dP = -\rho g dy = -g\,\frac{\overline{m}P}{RT} dy \,. \tag{6}$$

Now, dividing through by P, then

$$\frac{dP}{P} = -\frac{\overline{m}g}{RT} dy \,. \tag{7}$$

4 Now to integrate

Integrating (7), we obtain the indefinite integral:

$$\int \frac{dP}{P} = -\frac{\overline{m}g}{RT} \int dy\,. \tag{8}$$

As I said, we're using the approximation that T is a constant.

Now, here's the key part: To talk about the air pressure at height $y = h$, we need to integrate the weight of the air, starting h and going all the way up. But I don't know how to do that. So, I will integrate from height h to height $y = Y$, which is the height at which the air pressure is just one Pascal. This should be plenty high to get a good approximate result, since the air pressure at sea level P_0 is $101,325$ Pa (Pascals).

At this point we need a relationship between the pressure at $y = Y$ and P_0, which we'll get this way (the reason for this we'll see soon enough)

$$\int_{P_0}^{P(Y)} \frac{dP}{P} = -\frac{\overline{m}g}{RT} \int_0^Y dy\,, \tag{9}$$

which gives us

$$\ln P \Big|_{P_0}^{P(Y)} = -\frac{\overline{m}g}{RT} Y\,, \tag{10}$$

where we use that $P(Y) = 1$. So this simplifies to

$$\ln P_0 = \frac{\overline{m}g}{RT} Y\,, \tag{11}$$

where, because of (10), P_0 is unitless, though we must remember that we set in pascals. As I said before, we need to integrate starting at $y = h$ all the way up to $y = Y$:

$$\int_{P(h)}^{P(Y)} \frac{dP}{P} = -\frac{\overline{m}g}{RT} \int_h^Y dy\,, \tag{12}$$

which gives us

$$\ln P \Big|_{P(h)}^{1} = -\frac{\overline{m}g}{RT} (Y - h)\,. \tag{13}$$

This expands to

$$-\ln P(h) = -\frac{\overline{m}g}{RT} (Y - h) = -\ln P_0 + \frac{\overline{m}g}{RT} h\,, \tag{14}$$

where we used (11). And the next step is

$$\ln P(h) - \ln P_0 = -\frac{\overline{m}g}{RT} h\,. \tag{15}$$

Then,

$$\ln \frac{P(h)}{P_0} = -\frac{g\overline{m}}{RT} h\,. \tag{16}$$

On exponentiating both sides we get

$$P(h) = P_0 e^{-\frac{g\overline{m}}{RT}h} \,. \tag{17}$$

And this is the formula we wanted to derive.

5 What are they doing?

Okay, I introduced this height Y because it allows me to think in terms of pressure being about the weight of all the air above a certain point on the y-axis (height). However, some people may think of Y as adscititious! Regarding the limits of integration, this is what everyone else does, that I did not do:

$$\int_{P_0}^{P(y)} \frac{dP}{P} = -\frac{\overline{m}g}{RT}\int_0^y dy \,. \tag{18}$$

But how does one get the pressure of the air starting at $y = h$ and going up by integrating starting at $y = 0$ and going up to $y = h$? Starting at zero and going up to h seems to ignore the air above $y = h$!

So, this is my proposed fix. Set

$$\int_{P_0}^{P(y)} = \int_{P_0}^{P(Y)} - \int_{P(y)}^{P(Y)} \,. \tag{19}$$

Now the integral on the LHS makes sense because it is expressed as the difference of two integrals, each of which makes physical sense, in that they represent the total air pressures resulting from all the air above a certain point, which is what air pressure actually means. Yes, in that upward movement the stop at $y = Y$ is arbitrary, but it should fit well within the approximations we're making. Besides, in (19) the pressures at elevation Y cancel out. That is, in deriving (11) but using it in (14), means that we never needed to explicitly calculate Y, though that should be easy to do.

Anyway, as far as I'm concerned, (19) mathematically justifies the use of (18) and gives it a logical meaning. Perhaps the way to motivate all those "extra" steps I used in the integration section, would be to present (19) at the beginning of the integration section, with explanation.

6 Conclusion

In some derivations I've seen, people set the number of moles equal to unity, but I don't understand the logic of that. My other issue with the approximation I derived is that I used the Ideal Gas Law without much justification for doing so. I just wonder why people employ the Ideal Gas Law in this derivation without justifying why this can be done.

Lastly, my biggest complaint is how glibly we justify (if we mention it at all!) why we set the temperature to a constant.