The Hard-Sphere Collision Problem

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Abstract

The Hard-Sphere Collision Problem. We add that the collision is elastic.

1 Statement of the Problem

This Hard-Sphere Collision problem is rather special: the collision is elastic, the spheres are of the same mass m, and the same radius r. They are also undeformable (hence, 'hard') and smooth. So, to make the problem interesting, they will not collide head on, but will be offset by the impact parameter b, which measures the distance between line demarking the velocity of the incoming sphere's center and the center of the target sphere.

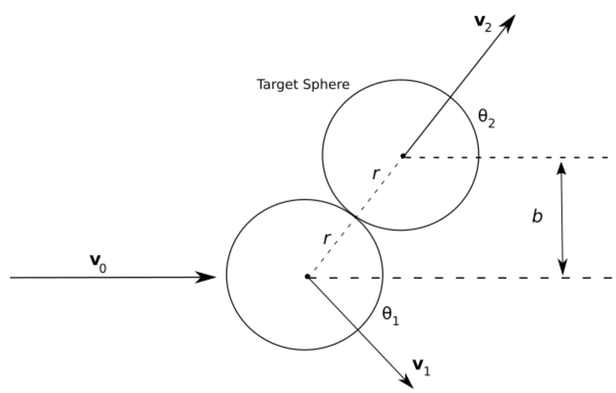

Figure 1. In the lab frame, the target sphere is at rest, and the moving sphere approaches it at velocity $\mathbf{v}_0$. In collisions, linear momentum is always conserved, but since this collision is also elastic, kinetic energy is also conserved. We'll set the x-axis in the direction of the incoming sphere, which will be along the bottom dashed line.

The kinetic energy of the system before collision is just $\frac{1}{2}mv_0^2$. So, because kinetic energy is conserved, we have that

$$\tfrac{1}{2}mv_0^2 = \tfrac{1}{2}mv_1^2 + \tfrac{1}{2}mv_2^2\,, \tag{1}$$

where v_0 is considered to be known. We add to this the equations of conservation of linear momentum along any two mutually orthogonal directions, which we'll choose conveniently as the x- and y-directions:

$$\text{momentum in } x\text{-direction:} \qquad mv_0 = mv_1 \cos\theta_1 + mv_2 \cos\theta_2 \,, \tag{2a}$$

$$\text{momentum in } y\text{-direction:} \qquad 0 = -mv_1 \sin\theta_1 + mv_2 \sin\theta_2 \,. \tag{2b}$$

Now, because the spheres are smooth, the only forces they can exert on each other are normal to their surfaces at the point of contact. That means that the force on the target sphere must be along the line connecting the two centers. Although this information does not give us the magnitude of $\mathbf{v}_2$, it does give us its direction, as determined by the angle θ_2, which we extract by the geometry of the collision:

$$\sin\theta_2 = \frac{b}{2r} \,. \tag{3a}$$

With a little algebra, we can calculate the cosine of the angle:

$$\cos\theta_2 = \left(1 - \frac{b^2}{4r^2}\right)^{1/2} \,. \tag{3b}$$

So, now there's nothing left to do analytically before we solve for variables v_1, v_2, and θ_1. Let's begin by simplifying (1):

$$v_0^2 = v_1^2 + v_2^2 \,. \tag{4}$$

Now, let's rewrite (2a) and (2b) into the forms

$$v_1 \cos\theta_1 = v_0 - v_2 \cos\theta_2 \,, \tag{5a}$$

$$v_1 \sin\theta_1 = v_2 \sin\theta_2 \,. \tag{5b}$$

Next, we square these:

$$v_1^2 \cos\theta_1^2 = v_0^2 - 2v_2 v_0 \cos\theta_2 + v_2^2 \cos^2\theta_2 \,, \tag{6a}$$

$$v_1^2 \sin\theta_1^2 = v_2^2 \sin^2\theta_2 \,. \tag{6b}$$

Adding these together, we get

$$v_1^2 = v_0^2 - 2v_2 v_0 \cos\theta_2 + v_2^2 \,. \tag{7}$$

On eliminating v_1 between (4) and (7) and simplifying, we get

$$v_2 = v_0 \cos\theta_2 = v_0 \left(1 - \frac{b^2}{4r^2}\right)^{1/2} \,. \tag{8}$$

We can now employ Eq. between (4) and (8) to solve for v_1, yielding

$$v_1 = \frac{v_0 b}{2r} \,. \tag{9}$$

Finally, we'll solve for θ_1 (actually, its sine) from (5b)

$$\sin\theta_1 = \frac{v_2}{v_1}\sin\theta_2\,. \tag{10}$$

Using (8), (9), and (3b), we get

$$\sin\theta_1 = \left(1 - \frac{b^2}{4r^2}\right)^{1/2}. \tag{11}$$

2 Conclusion

It's a simple matter to solve the adjusted problem in which the masses and/or radii of the two spheres are different. However, it will look a lot messier.