Cosmology Notes for L. Susskind's Lecture Series (2013), Lecture 1

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January 31, 2023

Abstract

This paper contains my notes on Lecture One of Leonard Susskind's 2013 presentation on Cosmology for his Stanford Lecture Series. These notes are meant to aid the viewer in following Susskind's presentation, without having to take copious notes. The fault for any inaccuracies in these notes is strictly mine.

1 Introduction

We could begin our discussion of cosmology in the distant past, but we'll skip that and begin with Newtonian physics, and build a cosmology based on Newton's equations, the total energy concept, and two empirical facts:

1. From the earth, the universe looks isotropic, meaning the same in all directions.
2. From the earth, the universe looks homogeneous, meaning the same at all points in space.

We will raise these observations to cosmological postulates. The second of these assumptions is less secure than the first because we cannot go far away from our solar system to get even a second observation on this. However, it is the purpose of the so-called *Cosmological Principle* to reject any assumption that we here on the earth have any preferred observation status, unless observation supports it. Accepting these two assumptions, we proceed. The CMBR (Cosmic Microwave Background Radiation) greatly supports the Cosmological Principle.

For our first analytical approximation, we shall regard the universe as a collection of point mass particles, each of which is a galaxy. There are approximately 10^{11} galaxies in the universe, and 10^{11} stars per galaxy (estimates as of 2013). The galaxies interact only by gravity.

The early guess at the stability of the universe was that it is static. We are going to use Newtonian mechanics to analyze just how reasonable that assumption was. The first thing needed to do is to introduce a suitable coordinate system.

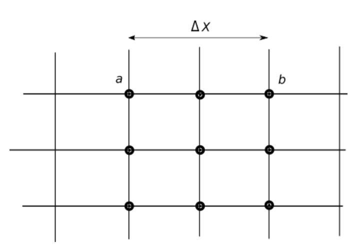

Figure 1. Our simplified version represents galaxies by the points on the grid intersections. The grid will follow the motion of the galaxies.

Referencing Fig. 1, if the galaxies are in motion (expansion or contraction), then the grid must keep up with this motion. The distance of galaxy a to galaxy b as seen from the in-between galaxy is given as

$$D_{ab} = a(t)\Delta X_{ab}\,. \tag{1}$$

Of course we must do our calculation in 3-dimensional space, and thus we get, more generally,

$$D_{ab} = a(t)\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}\,. \tag{2}$$

Differentiating this we get

$$V_{ab} = \dot{a}(t)X_{ab}\,. \tag{3}$$

On dividing (3) by (1), we get

$$\frac{V_{ab}}{D_{ab}} = \frac{\dot{a}(t)}{a(t)} \equiv H(t)\,, \tag{4}$$

which we will refer to as 'Hubble's Law'. Note that this formula treats points a and b as arbitrary. Anyway, this last equation can be rewritten as

$$V = HD\,. \tag{5}$$

Next, we define ν as the mass per unit volume of the grid. Therefore, in volume $\Delta x\Delta y\Delta z$, we get

$$M = \nu\Delta x\Delta y\Delta z\,, \tag{6}$$

with

$$V(t) = a^3\Delta x\Delta y\Delta z\,. \tag{7}$$

Now we define the density

$$\rho = \frac{\nu}{a^3}\,. \tag{8}$$

The mass in a given cell is fixed because the mass moves with the grid.

We lose nothing by setting our origin at the earth/sun position, which we can consider to be "at rest."

Now we consider a galaxy at D from our origin.

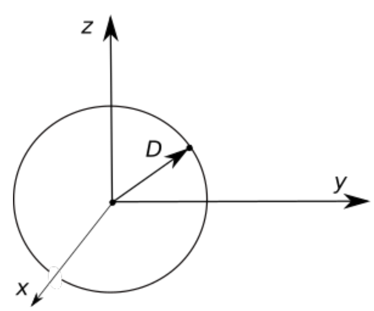

Figure 2. The galaxy at distance D 'rests' on the surface of a sphere of radius D and volume $4\pi D^3/3$.

It's well known in Newtonian theory of gravity that we calculate the gravitational force on the galaxy at D, when the mass is spherically symmetrically distributed about the origin, by 1) ignoring all the mass outside of the sphere and 2) by placing all the mass inside the sphere at the origin. The equations we need are

$$\begin{aligned} D &= a(t)\sqrt{x^2+y^2+z^2} = a(t)R\,,\\ V &= \dot{a}(t)R\,,\\ A &= \ddot{a}(t)R\,. \end{aligned} \tag{9}$$

As promised, we will analyze the force on the galaxy by Newtonian's method.

$$F = \frac{m_{\text{galaxy}} M_{\text{sphere}}}{D^2} G \,. \tag{10}$$

Then,

$$A = -\frac{M_{\text{sphere}} G}{D^2} = \ddot{a}(t) R \,. \tag{11}$$

From this we get

$$\frac{\ddot{a}}{a} = -\frac{M_{\text{sphere}} G}{a^3 R^3} \,. \tag{12}$$

Let's add to this the volume content:

$$\text{Vol}_{\text{sphere}} = \frac{4}{3}\pi D^3 = \frac{4}{3}\pi a^3 R^3 \,. \tag{13}$$

And finally,

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G \rho \,, \tag{14}$$

where we interpret ρ as the density of the universe at some appropriately large scale. From this equation we can deduce that the position of the origin is arbitrary. Equation (14) can be recast to

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi \frac{G\nu}{a^3} \,, \tag{15}$$

which was first described by Alexander Friedman [1888 – 16 Sept 1925].

So, we can talk about the acceleration of the universe, and its equation can be presented as

$$\ddot{x} = -\frac{MG}{x^2} \,, \tag{16}$$

though we cannot say what its future velocitiy will be without a better knoweldge of the masses involved.

Our next attack on the problem is to use energy considerations to determine the motion of the galaxy. We begin with the usual classical equation

$$E = \tfrac{1}{2} m v^2 - \frac{mMG}{x} \,. \tag{17}$$

Case 1) With $E < 0$, the galaxies must at some future time go to rest and then change direction of their motion.

Let's re-formulate the energy equation to

$$E = \tfrac{1}{2} \dot{a}^2 R^2 - \frac{mMG}{aR} \,. \tag{18}$$

Case 2) Setting E to zero, we get

$$\tfrac{1}{2} \dot{a}^2 R^2 - \frac{mMG}{aR} = 0 \,. \tag{19}$$

And using (13) we have the Friedman Equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G \rho \,. \tag{20}$$

Alternatively, with $\rho = \nu / a^3$,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi\nu}{3} \frac{G}{a^3} \,. \tag{21}$$

Hence, we can write

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{k^2}{a^3}\,, \tag{22}$$

where I introduced k as a mere constant of proportionality. On taking the square root of both sides and simplifying, we get.

$$\dot{a}a^{1/2} = k\,. \tag{23}$$

Multiplying through by dt and integrating, we get

$$\int a^{1/2} da = k \int dt\,, \tag{24}$$

yielding

$$a^{3/2} = \tfrac{3}{2}kt + c'\,. \tag{25}$$

On setting $c' = 0$, then

$$a = ct^{2/3}\,. \tag{26}$$