Cosmology Notes for L. Susskind's Lecture Series (2013), Lecture 10

P. Reany

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Abstract

This paper contains my notes on Lecture Ten of Leonard Susskind's 2013 presentation on Cosmology for his Stanford Lecture Series. These notes are meant to aid the viewer in following Susskind's presentation, without having to take copious notes. The fault for any inaccuracies in these notes is strictly mine.

1 Fundamental Quantities and Dimensional Analysis

The fundamental quantities of physics:

1. Speed of light c, universal local speed limit.
2. Planck's constant $\hbar$, universal uncertainty of simul. position and momentum measurements.
3. Gravitational constant G, universal mutual attraction between matter/energy.

6 dimensional units:

$c \qquad L,$

$\hbar \qquad t,$

$G \qquad m.$

The fundamental measurement units are length, time, and mass: $\ell,\ t,\ m$

Expressing the fundamental quantities of physics in terms of their fundamental units, we have:

1. $[\,c\,] = \ell/t\,,$
2. $[\,\hbar\,] = \ell^2 m/t\,,$
3. $[\,G\,] = \ell^3 m/^2 t\,.$

If we analyze the acceleration equation for Newtonian acceleration under the gravitational central force, we get

$$[\,a\,] = \left[\frac{MG}{r^2}\right], \tag{1}$$

which gives

$$\frac{\ell}{t^2} = \frac{m\,[\,G\,]}{\ell^2}\,. \tag{2}$$

On solving for $[\,G\,]$, we get

$$[\,G\,] = \frac{\ell^3}{mt^2}\,. \tag{3}$$

Now we define a thing called the 'Planck length' ℓ_P:

$$\begin{aligned}\ell_P &= [\,c\,]^a\,[\,\hbar\,]^b\,[\,G\,]^c \\ &= \frac{\ell^a}{t^a}\frac{\ell^{2b}m^b}{t^b}\frac{\ell^{3c}}{m^c t^{2c}}\,.\end{aligned} \tag{4}$$

Now, to cancel out the mass unit, set $c = b$, leaving us with

$$\ell_P = \frac{\ell_P^a}{t^a}\frac{\ell_P^{2b}}{t^b}\frac{\ell_P^{3c}}{t^{2c}}\,. \tag{5}$$

To get rid of time, we need to set $a + b + 2b = 0$. And to force the RHS to reduce to just ℓ_P, we need to set $a + 5b = 1$. On solving this sytem of constraints, we get

$$\begin{aligned}b = c &= \tfrac{1}{2}\,, \\ a &= -3/b\,.\end{aligned} \tag{6}$$

Substituting into (5), we get for ℓ_P^2

$$\ell_P^2 = [\,c\,]^{2a}\,[\,\hbar\,]^{2b}\,[\,G\,]^{2c} = \frac{\hbar G}{c^3}\,. \tag{7}$$

Hence,

$$\ell_P = \sqrt{\frac{\hbar G}{c^3}} \approx 10^{-35}\ \text{meters}\,. \tag{8}$$

How long will it take for light to travel a Planck distance?

$$t_P = \frac{d}{c} = \frac{10^{-35}\text{m}}{3\times 10^8\,\text{ms}^{-1}} \approx 10^{-43}\ \text{sec}\,, \tag{9}$$

which is called the "Planck time." Now, for the Planck mass m_P, which we develop from the relation $\Delta E \Delta t \sim \hbar$:

$$(m_P c^2)(t_P) \sim \hbar\,. \tag{10}$$

$$m_P = \frac{\hbar}{c^2 t_P} \sim 10^{-8}\ \text{kg}\,. \tag{11}$$

At the scale of the LHC (Large Hadron Collider), the Standard Model looks correct. The fine-tuned parameters can't be explained on the basis of the Standard Model of particle physics.

The Planck mass can be thought of as the mass of the simplest possible black hole. To switch to "Planck units" we could set $c = \hbar = G = 1$.

2 Map of the universe

The farthest back we can see in the universe is the CMB, which is red-shifted by about a thousand, with temperature about 2.7 K.

We infer that the early universe must have had some inhomogeneity.

$$\frac{\delta T}{T} \sim 10^{-5}\ \text{in the CMB}\,. \tag{12}$$

1) Thus 10^{-5} is a calculated value of the analytics via computer simulations to arrive at the known mass distribution of the universe.

2) 10^{-5} is the direct measurement of the CMB. Temperature fluctuations come form mass flutuations locally, the lower the temperature. $10^{-5} \sim \delta\rho/\rho$ at the time of decoupling.

There is a pattern in the CMB that is scale invariant.

We need two big principles to understand the origin of the inhomogeneity in the universe.

1. The quantum zero-point oscillation of a harmonic oscillator.
2. The damped harmonic oscillator.

$$E = \frac{\dot{x}^2}{2} + \frac{\omega^2 x^2}{2}\,. \tag{13}$$

The time average sets

$$\langle\,\mathrm{KE}\,\rangle = \langle\,\mathrm{PE}\,\rangle\,. \tag{14}$$

Quantum mechanically,

$$E = (n + \tfrac{1}{2})\hbar\omega \tag{15}$$

counts up how many units of excitation there are.

Ground state energy

$$E = \tfrac{1}{2}\hbar\omega\,. \tag{16}$$

Therefore, on the average

$$\omega^2 x^2 = \tfrac{1}{2}\hbar\omega\,. \tag{17}$$

So,

$$x \approx \sqrt{\hbar/\omega} \tag{18}$$

is the approximate range of oscillation. Thus,

$$x(t) \approx \sqrt{\hbar/\omega}\, e^{i\omega t} \tag{19}$$

and

$$\dot{x}(t) \approx \sqrt{\hbar\omega}\, i\, e^{i\omega t} \tag{20}$$

Thus

$$\dot{x}^2 \approx \hbar\omega\,. \tag{21}$$

An equation of motion

$$\ddot{x} + x\omega^2 = 0 \tag{22}$$

Or,

$$\ddot{x} = -x\omega^2 \tag{23}$$

where ω acts as a spring constant and $-x\omega$ is the restoring force.

But damped harmonic motion needs viscosity.

$$\ddot{x} + \gamma\dot{x} + x\omega^2 = 0 \tag{24}$$

For a solution, try $x \sim e^{\alpha t}$, then

$$\dot{x} = \alpha e^{\alpha t} \tag{25}$$

$$\ddot{x} = \alpha^2 e^{\alpha t} \tag{26}$$

Hence,

$$\alpha^2 + \gamma\alpha + \omega^2 = 0\,, \tag{27}$$

which has possible solutions

$$\alpha = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega^2}}{2} = 0\,. \tag{28}$$

For the case $\gamma^2 < 4\omega^2$ we have weak viscosity.

$$\alpha = -\frac{\gamma}{2} \pm \frac{i}{2}\sqrt{4\omega^2 - \gamma}\,. \tag{29}$$

And

$$x = ce^{-\frac{\gamma}{2}t} e^{\pm\frac{i}{2}\sqrt{4\omega^2 - \gamma^2}\,t}\,. \tag{30}$$

For γ small, this is a damped, oscillatory curve.

For the case $\gamma^2 > 4\omega^2$ we have the over-damped case.

$$\alpha_{\pm} = -\frac{\gamma}{2} \pm \tfrac{1}{2}\sqrt{\gamma^2 - 4\omega^2}\,, \tag{31}$$

and $\alpha_{\pm} < 0$. No oscillation.

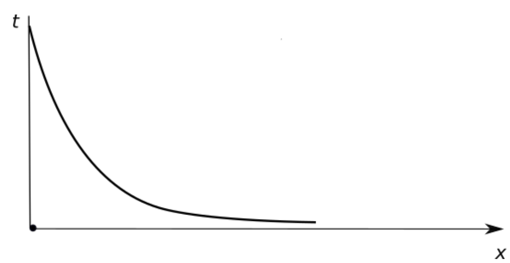

Figure 1.Over-damped, no oscillation.

For the case $\gamma^2 = 4\omega^2$ we have critical damping, the crossover case.

$$\alpha_{\pm} = -\frac{\gamma}{2}\,, \tag{32}$$

and $\alpha_{\pm} < 0$. No oscillation.

If the restoring force is zero,

$$\alpha^2 + \gamma\alpha = 0\,, \tag{33}$$

or

$$\alpha(\alpha + \gamma) = 0\,, \tag{34}$$

Subcases: 1) $\alpha = 0$ or 2) $\alpha = -\gamma$

Case $\alpha = 0$ then $c_0 e^{\alpha t} = c_0$ and $c_1 e^{-\gamma t}$. So, the solution is a linear combination: $c_0 + c_1 e^{-\gamma t} \to c_0$.

Now, ω is a function of time.

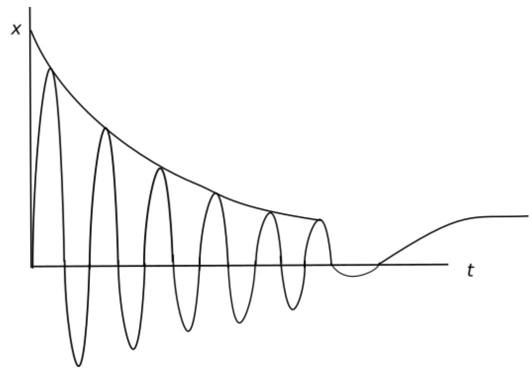

Figure 2. If ω get sufficiently small, the oscillations stop.

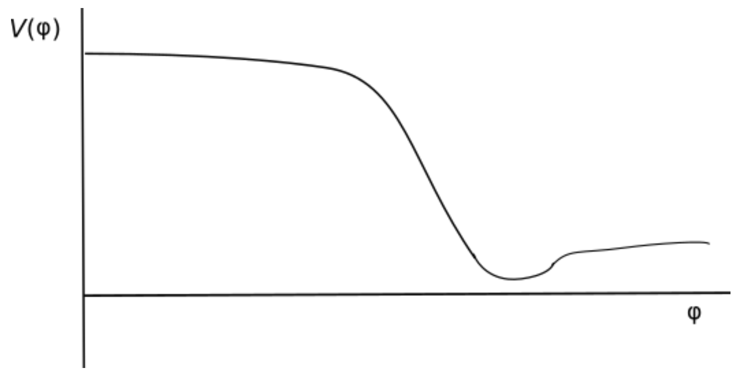

Figure 3. The Potential Energy Density V (the 'Inflaton field') is a slowly decreasing scalar field.

We make the approximation that $V'(\varphi) \approx 0$. In flat spacetime

$$\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} = 0\,, \tag{35}$$

gives us the wave equation.

In the background of an expanding universe, we have that

$$\frac{\partial^2 \varphi}{\partial t^2} + 3H\frac{\partial \varphi}{\partial t} = 0\,, \tag{36}$$

If the scalae factor is constant, we should get (35). Hence, (36) is lacking an important term.

$$\frac{\partial^2 \varphi}{\partial t^2} + 3H\frac{\partial \varphi}{\partial t} - \frac{1}{a^2(t)}\frac{\partial^2 \varphi}{\partial x^2} = 0\,, \tag{37}$$

Note: $a\Delta x$ is an actual physical distance. Anyway, we'll attempt a solution in the form of $\phi_k(t)e^{ik\cdot x}$.

$$\ddot{\phi} + 3H\dot{\phi} + \frac{k^2}{a^2(t)}\phi_k(t) = 0\,, \tag{38}$$

and we're back to the damped harmonic oscillator. This equation has a pattern similar to that in Fig. 2.

After the expansion that smoothed out the universe, there is more to the story. We need to include the effects of quantum mechanics in all this. There is a zero-point oscillation in space for every value of k. Ordinarily, these oscillations can go unnoticed because these frequencies of oscillation are very large.

At what time does the critical damping happen? The answer depends on k. The smaller the wavelength, the later the oscillation will stop.

$$\gamma = 2\omega \,, \tag{39}$$

but $\gamma = 3H$ and $\omega = k/a$, therefore,

$$3H = 2\frac{k}{a} \,. \tag{40}$$

Therefore the transition from under-damped to over-damped occcurs when

$$a = \frac{2k}{3H} \sim \frac{k}{H} \,. \tag{41}$$

Now,

$$\lambda = \frac{2\pi}{k} \quad \text{or} \quad k = \frac{2\pi}{\lambda} \,. \tag{42}$$

However, same old problem here. This λ is in the artificial units of the expansion. To get λ in units of actual physical lengths, we need to divide it by a, thus,

$$k = \frac{2\pi a}{\lambda} \,. \tag{43}$$

So, to an approximation,

$$a = \frac{a}{\lambda H} \,. \tag{44}$$

Thus, the crossover happens when

$$\lambda = \frac{1}{H} \,. \tag{45}$$

By definition, $v = HD$. So, for $v = c = 1$, $D = \frac{1}{H}$, which gives the distance at which galaxies are receding at the speed of light (the Hubble Horizon). So, these wavelengths are expanding along with the expansion of the universe, as given by $a(t)$. But our interest here is the expansion period during the inflation episode. At this time, the expansion was so fast that the Hubble horizon was microscopic. Once any wavelength is expanded to the Horizon size, the oscillation stops and the critical damping begins. (The restoring force goes to zero.)

But there's more. For each rapidly succeeding Horizon comes a new set of waves, and the whole thing repeats itself. These new waves come from the vacuum fluctuations. All this accumulative waves ramming through this universe causes the inhomogeneity in the universe.