Cosmology Notes for L. Susskind's Lecture Series (2013), Lecture 2

P. Reany

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Abstract

This paper contains my notes on Lecture Two of Leonard Susskind's 2013 presentation on Cosmology for his Stanford Lecture Series. These notes are meant to aid the viewer in following Susskind's presentation, without having to take copious notes. The fault for any inaccuracies in these notes is strictly mine.

1 Review

Last time we introduced the density variable

$$\rho = \frac{\nu}{a^3}\,. \tag{1}$$

The mass in a given cell is fixed because the mass moves with the grid. But this variable lack physical meaning at this point because we have not specified a unit distance for the grid. However, we can make some sense out of appropriate ratios of functions of a, which we defined implicitly by

$$D_{ab} = a(t)\Delta X_{ab}\,, \tag{2}$$

where a and b are points on the grid. Differentiating this we get

$$V_{ab} = \dot{a}(t)X_{ab}\,. \tag{3}$$

On dividing (3) by (2), we get

$$\frac{V_{ab}}{D_{ab}} = \frac{\dot{a}(t)}{a(t)} \equiv H(t)\,, \tag{4}$$

which we will refer to as 'Hubble's Law'. Note that this formula treats points a and b as arbitrary. Anyway, this last equation can be rewritten as

$$V = HD\,. \tag{5}$$

Employing energy considerations, we obtained the Friedman Equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\frac{G\nu}{a^3}\,. \tag{6}$$

We can simplify this last equation by making a suitable lumping together of constants by a suitable fixing of ν, to get

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{a^3}\,, \tag{7}$$

whose solution takes the form

$$a \sim t^{2/3}\,. \tag{8}$$

The equation models a decelerating universe that, nevertheless, never comes to rest.

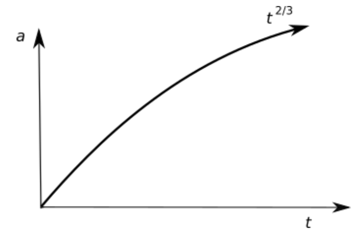

Figure 1. Our crude modelso far shows the universe expanding forever.
But it's also decelerating forever.

2 Moving On

We return to our classical energy equation with energy a constant.

$$E = \tfrac{1}{2}mv^2 - \frac{mMG}{D}\,. \tag{9}$$

Then, dividing by m,

$$\frac{E}{m} = \tfrac{1}{2}v^2 - \frac{MG}{D}\,. \tag{10}$$

Or, rather,

$$\frac{2E}{m} = v^2 - \frac{2MG}{D}\,. \tag{11}$$

But we'll begin at $x = 1$.

$$D = ax = a\,, \tag{12a}$$

$$V = \dot{a}x = \dot{a}\,. \tag{12b}$$

Hence,

$$\frac{2E}{m} = \dot{a}^2 - \frac{2MG}{a} \equiv c^2\,, \tag{13}$$

for some constant c and for constants of energy and mass. (Note: I used c^2 instead of Susskind's c for later purposes.)

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{2MG}{a^3} = \frac{c^2}{a^2}\,. \tag{14}$$

But

$$V = \frac{4}{3}\pi a^3\,. \tag{15}$$

Therefore,

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi}{3}\frac{MG}{V} = \frac{c^2}{a^2}\,. \tag{16}$$

Or, using ρ:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho + \frac{c^2}{a^2}\,. \tag{17}$$

This last equation is the Friedman Equation for nonzero energy. If $\dot{a} > 0$ then the universe will expand forever.

Right now, we'll consider asymptotic $a \to \infty$:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{c^2}{a^2}\,, \tag{18}$$

which has the solution

$$a = ct\,. \tag{19}$$

This makes sesnse. Eventually, the pullback on a particular galaxy by gravity will tend to zero as the universe expands, tending to leave the speed of recession at a constant value. This is referred to as the *Matter Dominated Universe*.

In the case of negative energy, as t goes larger, at some point

$$\frac{8\pi}{3}\frac{G\nu}{a^3} - \frac{c^2}{a^2} < 0\,, \tag{20}$$

resulting in the cosmic bump curve below.

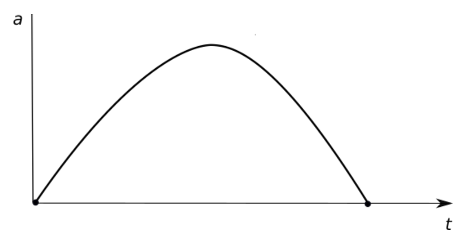

Figure 2. With negative energy, what goes out must come back.

3 Relation to Einstein's GR

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{c^2}{a^2} = \frac{8\pi}{3}G\rho\,. \tag{21}$$

In this form, the LHS describes geometry and the RHS describes energy-momentum. However, our preferred form to analyze will be

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{\kappa}{a^2} = \frac{8\pi G}{3}\rho\,. \tag{22}$$

It's time to include radiation in the energy-momentum term. We now begin with a cube of unit dimension on all sides. Then

$$V = a^3\,. \tag{23}$$

We assume that the number of photons is fixed, but that their energy decreases in time due to cosmic inflation. (By a process not well explained.)

$$E = \frac{h}{\lambda}\,. \tag{24}$$

So, now we look at the *Radiation Dominated Universe* equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\frac{\overline{\nu}}{a^4} - \frac{c}{a^2}\,. \tag{25}$$

Then, for early times, when the first term on the RHS of (25) dominates over the second term, we get

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\frac{\overline{\nu}}{a^4}\,, \tag{26}$$

Similer to our previous simplification, we choose $\overline{\nu}$ so that

$$\frac{8\pi G}{3}\overline{\nu} \to 1\,. \tag{27}$$

So we get

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{a^4}\,, \tag{28}$$

which has the solution

$$a \sim t^{1/2}\,. \tag{29}$$

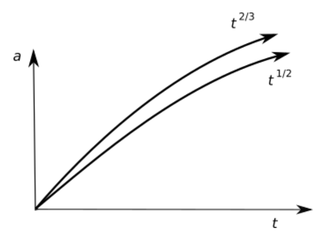

Figure 3. Comparison of matter-dominated to energy-dominated.

Now, for the mixed case.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{C_M}{a^3} + \frac{C_R}{a^4}\,. \tag{30}$$

For small a, the radiation term dominates, but eventually the matter term will dominate. This was the state of cosmology up to about the 1980s.

At this point, we introduce the cosmological curvature constant κ:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}\,. \tag{31}$$

Then,

1. $k < 0$, negatively curved space
2. $k = 0$, flat space
3. $k > 0$. spherical geometry