Cosmology Notes for L. Susskind's Lecture Series (2013), Lecture 3

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Abstract

This paper contains my notes on Lecture Three of Leonard Susskind's 2013 presentation on Cosmology for his Stanford Lecture Series. These notes are meant to aid the viewer in following Susskind's presentation, without having to take copious notes. The fault for any inaccuracies in these notes is strictly mine.

1 Review

Core beliefs: the universe is homogeneous, isotropic, and maybe flat. Maybe flat? Susskind wanted to keep the question of the curvature of space open because he proposed that space may be toriordal. What are the possible geometries of the universe if we maintain that the universe is homogeneous?

Anyway, let's look at the line element (metric) for space, beginning with one for a circle in 3D in flat space

$$ds^2 = dx^2 + dy^2 + dz^3 \,. \tag{1}$$

Now, let's make a shift from rectangular to polar coordinates.

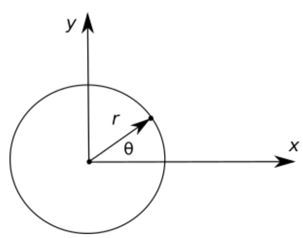

Figure 1. Typical conversion of rectangular to polar coordinates.

The circle is the 1-sphere, or Ω_1

$$\begin{aligned} ds^2 &= dx^2 + dy^2 \\ &= dr^2 + r^2 d\theta^2 \\ &= dr^2 + r^2 d\Omega_1^2 \,. \end{aligned} \tag{2}$$

We have a recursive deifnition of Ω on higher-dimensional spheres. On a unit circle, the metric becomes just $ds^2 = d\Omega_1^2$.

Next, the sphere (2-sphere) is also a homogeneous surface, being the same at every point. So, we pick a point P on the sphere and seek to survey the sphere in a radially systematic way. How?

By foliating the sphere as a set of concentric circles, starting at P and moving away until the last circle of radius zero terminmates at the point antipodal to P.

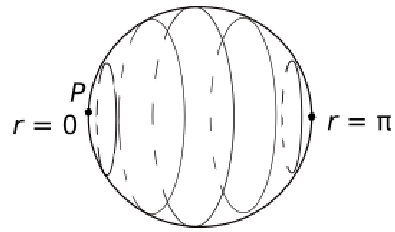

Figure 2. The astronomer at $r = 0$ sees the universe as circles on the sphere. The point at $r = \pi$ is the farthest away we can see.

Now, the radius of each circle goes as sine of r. Hence,

$$ds^2 = dr^2 + \sin^2 r\, d\theta^2 = dr^2 + \sin^2 r\, d\Omega_1^2\,, \tag{3a}$$

$$d\Omega_2^2 = dr^2 + \sin^2 r\, d\Omega_1^2\,, \quad \text{[new name for 2-sphere]} \tag{3b}$$

$$d\Omega_3^2 = dr^2 + \sin^2 r\, d\Omega_2^2\,. \quad \text{[new name for 3-sphere]} \tag{3c}$$

I've already stated Ω_3, but let's try to visualize it.

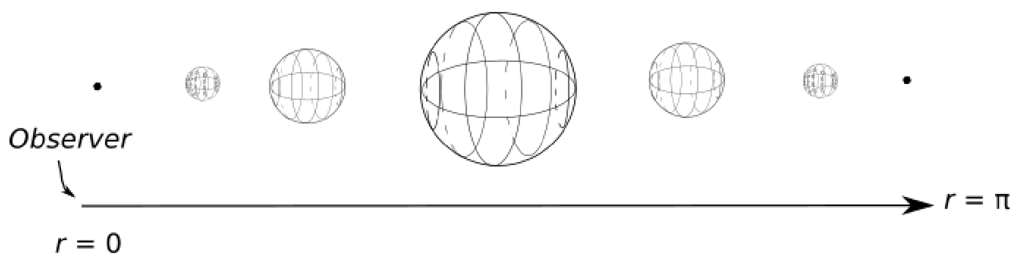

Figure 3. The astronomer at $r = 0$ sees the universe as concentric spheres, which are at first expanding, but then contracting.

By the way, for flat 3-dimensional space, we have that

$$ds^2 = dr^2 + r^2\, d\Omega_1^2\,. \tag{4}$$

We know how to embed the 1-dimensional circle into two dimensions:

$$x^2 + y^2 = 1\,, \tag{5}$$

for a unit circle. Similarly, for the 2-sphere in 3-dimensions:

$$x^2 + y^2 + z^2 = 1\,. \tag{6}$$

Next, we generalize to the unit 3-sphere in 4-dimensions:

$$x^2 + y^2 + z^2 + w^2 = 1\,. \tag{7}$$

Suppose we have a telescope that allows us to determine the distance to a galaxy in flat space. Our question is about the angle that such a galaxy subtends in the sky. For simplicity, we will assume that all galaxies have the same diameter d.

$$ds^2 = d^2 = r^2\, d\theta^2\,. \tag{8}$$

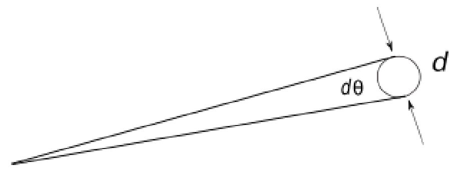

Figure 4. The angle subtended of a distant galaxy of arc length d.

From (8) we can solve for $d\theta$:

$$d\theta = \frac{d}{r}\,. \tag{9}$$

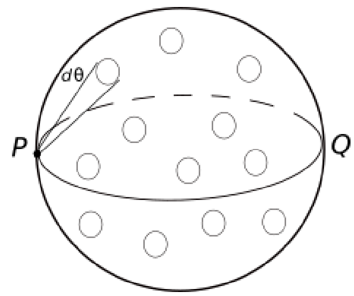

Figure 5. Galaxies on the sphere. The point P is the observer's position in space.

From (3a) we can solve for $d\theta$:

$$d\theta = \frac{d}{\sin r}\,. \tag{10}$$

Hence, the galaxies near to P and Q look bigger than those in between them. A galaxy at Q would fill the sky.

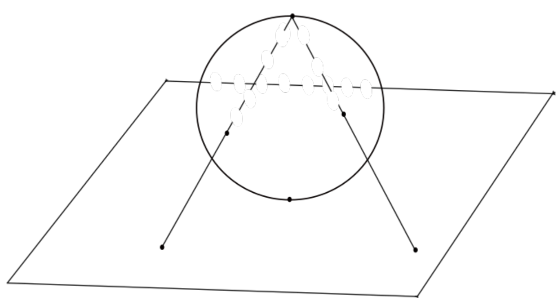

Figure 6. Stereographic Projection. The sphere has North and South Poles. The observer is at the South Pole. Points on the sphere are projected onto points on the plane.

In stereographic projection, circles are mapped to circles, those nearest the north pole being magnified the most. But those nearest the south pole are distorted the least. It's also possible to project 3-spheres. The point at the north pole maps to a circle of infinite radius. If we imagine that little circles on the sphere are galaxies, they get mapped to ever larger circles on the plane as the galaxies move to the north pole.

Returning to Eq. (3b), with the substitution $\Omega \to \mathcal{H}$, where $\mathcal{H}$ stands for 'hyperbolic', we have that

$$d\mathcal{H}_2^2 = dr^2 + \sinh^2 r\, d\Omega_1^2\,, \tag{11}$$

Let's look for a moment at the hyperbolic sine function.

$$\sinh r \equiv \frac{e^r - e^{-r}}{2}\,. \tag{12}$$

Thus, for large r, $e^{-r} \to 0$, and $\sinh r \to e^r/2$. So, in this space circles grow in size rapidly as r goes large. On going to three-dimensions,

$$d\mathcal{H}_3^2 = dr^2 + \sinh^2 r\, d\Omega_2^2\,, \tag{13}$$

which describes the behavior of hyperbolic two-spheres.

2 Hyperboloids

Referring again to Fig. 4,

$$d^2 = \sinh^2 r\, d\theta^2\,. \tag{14}$$

Therefore,

$$d\theta = \frac{d}{\sinh r}\,, \tag{15}$$

which, for large r becomes

$$d\theta = 2d e^{-r}\,. \tag{16}$$

Under these conditions, the angle subtended by this galaxy would go to zero quickly as r goes large, while the number of galaxies must grow fast. For a model of hyperbolic space, consider Escher's hyperbolic disk of angels and demons.

The following equation describes an hyperboloid

$$T^2 - x^2 - y^2 = 1\,, \tag{17}$$

where the radius is the value on the RHS, in this case, being unity. The following figure graphs this equation:

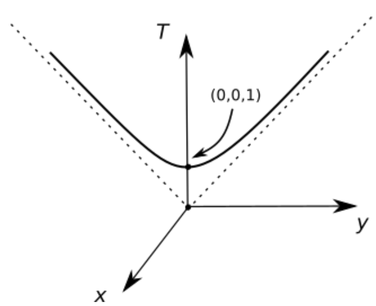

Figure 6. All points on this hyperboloid are equivalent. Spin the dotted line about the τ-axis to get a cone. Spin the hyperbola likewise to get a hyperboloid.

Deriving the differential line element from the graph in Fig. 6, we get

$$ds^2 = dx^2 + dy^2 - d\tau^2\,, \tag{18}$$

which makes all points on the hyperboloid equivalent. (Performs a Lorentz transformation.)

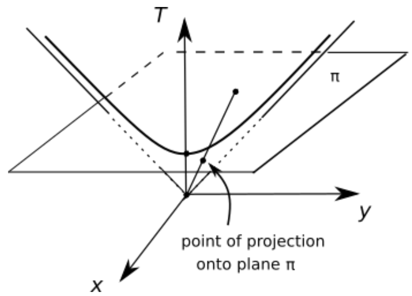

Figure 7. The hyperboloid intersects the plane π in a circle (not shown). Every point of the hyperboloid projects onto plane π inside this circle.

The line element for the sphere is

$$ds^2 = a^2(dr^2 + \sin^2 r d\Omega^2)\,. \tag{19}$$

The line element for the hyperbolic cone is is

$$ds^2 = a^2(dr^2 + \sinh^2 r d\Omega^2)\,. \tag{20}$$

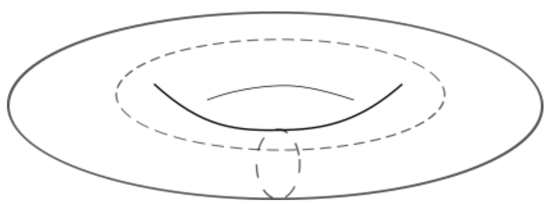

Figure 8. It's possible that the universe is like a huge torus.

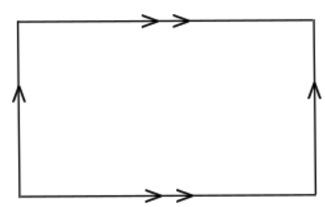

Figure 9. Topologically, the torus can be represented by identifying the opposite sides of a rectangle.

In special relativity, light rays are null rays.

$$ds^2 = 0 = -dt^2 + dx^2\,, \tag{21}$$

or

$$dx = \pm dt\,. \tag{22}$$

Now, let's generalize the 2-sphere.

$$ds^2 = -dt^2 + a^2 d\Omega_2^2\,, \tag{23}$$

where $a(t)$ is the radius of the sphere. For an n-sphere, (25) becomes

$$ds^2 = -dt^2 + a^2 d\Omega_n^2\,, \tag{24}$$

Two distinct points on a sphere, subtended by angle θ, will have "distance"

$$D = a\theta\,, \tag{25}$$

The relative velocity of these two points due to expansion is

$$V = \dot{a}\theta \qquad (\theta\text{ fixed})\,. \tag{26}$$

Thus

$$\frac{V}{D} = \frac{\dot{a}}{a}\,, \tag{27}$$

and

$$H = \frac{\dot{a}}{a}\,, \tag{28}$$

This is true for both the flat plane and the hyperboloid.

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)\,, \tag{29}$$

which depicts a flat-world grid but the separation is a function of $a(t)$.

In the spherical case:

$$ds^2 = -dt^2 + a^2(t)d\Omega_3^2\,, \tag{30}$$

and in the hyperbolic case:

$$ds^2 = -dt^2 + a^2(t)d\mathcal{H}_3^2\,, \tag{31}$$