Cosmology Notes for L. Susskind's Lecture Series (2013), Lecture 4

P. Reany

February 2, 2023

Abstract

This paper contains my notes on Lecture Four of Leonard Susskind's 2013 presentation on Cosmology for his Stanford Lecture Series. These notes are meant to aid the viewer in following Susskind's presentation, without having to take copious notes. The fault for any inaccuracies in these notes is strictly mine.

1 Review

Let's look at the homogeneity of space from the viewpoint of the metric. If we start off in a flat space in 2-d in coordinates x_1 and x_2, we have the line element. Anyway, let's look at the line element (metric) for flat space

$$ds^2 = dx_1^2 + dx_2^2 \,. \tag{1}$$

If we then change coordinates to y_1 and y_2, given by

$$\begin{aligned} y_1 &= x_1 + a_1 \,, \\ y_2 &= x_2 + a_2 \,, \end{aligned}$$

then the line element becomes

$$ds^2 = dy_1^2 + dy_2^2 \,, \tag{2}$$

which has the same form as (1). Now, if there exists a translation of coordinates that take the origin to any other point, leaving the metric the same, the space is said to be 'homogeneous'.

Consider the case of the sphere.

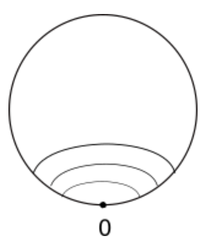

Figure 1. We begin with the origin at the bottom of the sphere.

$$ds^2 = dr^2 + \sin^2 r \, d\theta^2 \,. \tag{3}$$

But we can make an arbitrary shift of origin by change of coordinates, to get

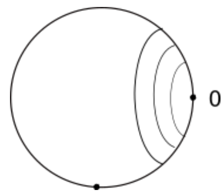

Figure 2. We shift of origin by change of coordinates.

After this change of coordinates, we get the new line element.

$$ds^2 = dr'^2 + \sin^2 r'\, d\theta'^2 \,. \tag{4}$$

A similar result would be true for hyperbolic coordinates. Every point on the sphere is the same as every other point on the sphere. The torus, though homogeneous, is not isotropic.

2 Introducing general relativity

We'll need the line element for the various geometries.

$$ds^2 = -dt^2 + a^2(t) \times \begin{cases} d\Omega_3^2 = dr^2 + \sin^2 r\, d\Omega_2^2, & k = 1\,, \\ dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\Omega_2^2, & k = 0\,, \\ dr^2 + \sinh^2 r\, d\Omega_2^2\,, & k = -1\,. \end{cases} \tag{5}$$

The distance D between fixed points in space is given by

$$D \propto a(t)\Delta\theta\,. \tag{6}$$

On differentiating, we get

$$V \propto \dot{a}(t)\Delta\theta\,. \tag{7}$$

And, on taking their ratio,

$$V \propto \frac{\dot{a}}{a}\,, \tag{8}$$

which doesn't depend on where you are.

Our cosmological Einstein equation is given as

$$R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R = \frac{8\pi G}{3} T^{\mu\nu}\,. \tag{9}$$

We are interested in the T^{00} component. $T^{00} = \rho$.

As it turns out, $R^{00} - \frac{1}{2}g^{00}R$ will have only the squares of first-order derivatives. Otherwise, this quantity will give us the curvature of the space.

Now, the time dependency of the line element exists only in $a(t)$, which yields

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G}{3}\rho \quad (k = -1, 0, 1)\,. \tag{10}$$

First, we reorganize:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \quad (k = -1, 0, 1)\,. \tag{11}$$

So, what about this ρ variable? It will comprise the rest mass of the objects in the region of interest. Now, all we need to continue is to find some relationship between ρ and $a(t)$. First, we will assume two relationships and then investigate their consequences. For a Matter-Dominated universe, we have that

$$\rho = \frac{\rho_0}{a^3}\,, \tag{12}$$

where ρ_0 is the value of ρ when $a = 1$. Next, we have for the Radiation-Dominated universe,

$$\rho = \frac{\rho_0}{a^4}\,. \tag{13}$$

Next, we have a two-step procedure to follow.

Step 1) Assume some form of an equation of state that relates pressure P and density ρ.

Step 2) Derive for different kinds of materials the equation of state.

We will postulate a relation between P and ρ along thermodynamic lines:

$$P = w\,\rho\,, \tag{14}$$

where w is a constant.

Now, for the case of the Matter-Dominated universe, $P = 0$ and therefore $w = 0$. For the case of the Radiation-Dominated universe, $w = 1/3$ and therefore $P = \frac{1}{3}\rho$.

So, consider a box of volume V. Taking ρ to be the energy density, then we have that

$$E = \rho V \tag{15}$$

is the energy in the box. If we consider what would happen if we changed the volume of the box by a differential amount dV, we get from the first law of thermodynamics that

$$dE = -P\,dV\,. \tag{16}$$

On the other hand, expanding (15) by differentials, would give us

$$dE = \rho dV + V d\rho\,. \tag{17}$$

After equating these last two equations and using separation of variables, we get

$$\frac{d\rho}{\rho} = -\frac{(1+w)dV}{V}\,. \tag{18}$$

Integrating this, we get

$$\rho = \frac{c}{V^{1+w}}\,, \tag{19}$$

where c is a constant of integration. But $V \propto a^3$, hence

$$\rho = \frac{c'}{a^{3(1+w)}}\,. \tag{20}$$

For the case $w = 0$ (the Matter-dominated case), $\rho \propto a^{-3}$. For the case $w = \frac{1}{3}$, $\rho \propto a^{-4}$. Therefore in our solutions, w is the key thing to determine.