Cosmology Notes for L. Susskind's Lecture Series (2013), Lecture 5

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Abstract

This paper contains my notes on Lecture Five of Leonard Susskind's 2013 presentation on Cosmology for his Stanford Lecture Series. These notes are meant to aid the viewer in following Susskind's presentation, without having to take copious notes. The fault for any inaccuracies in these notes is strictly mine.

1 Review

First, we will assume two relationships and then investigate their consequences. For a Matter-Dominated (MD) universe, we have that

$$\rho = \frac{\rho_0}{a^3}\,, \tag{1}$$

where ρ_0 is the value of ρ when $a = 1$. Next, we have for the Radiation-Dominated (RD) universe,

$$\rho = \frac{\rho_0}{a^4}\,. \tag{2}$$

Now, we add to these last two equations an equation of state (EOS) that will let us connect ρ to $a(t)$. For pressure P we assume the relation

$$P = w\rho\,, \tag{3}$$

where w is a number that characterizes a 'fluid'. Recall that in the MD universe $P = 0$, since the particles (galaxies) move slowly and interact little. Let's consider the pressure in a box with a gas at equilibrium. If we change the volume of the box slightly, then work is done:

$$dW = F\,dx\,, \tag{4}$$

where dx is the direction in which the change of volume is made. Then

$$dW = F\,dx = PA dx = PdV\,. \tag{5}$$

Therefore,

$$dE = -PdV\,. \tag{6}$$

But we also know that

$$E = \rho V\,. \tag{7}$$

Hence,

$$dE = \rho dV + V d\rho = -PdV = -w\rho dV\,. \tag{8}$$

Simplifying, we get

$$V d\rho = -(1+w)\rho dV \,. \tag{9}$$

Then, after separation of variables:

$$\frac{d\rho}{\rho} = -(1+w)\frac{dV}{V} \,. \tag{10}$$

After integrating,

$$\rho = \frac{c'}{V^{1+w}} = \frac{c}{a^{3(1+w)}} \,, \tag{11}$$

where we used that the volume is proportional to the scale factor. In conclusion:

MD: $P \approx 0 \quad \Rightarrow w = 0, \quad \rho \sim 1/a^3$.

RD: $\quad w = 1/3 \quad 1 + w = 4/3, \quad \rho \sim 1/a^4$.

2 Proof of $P = \frac{1}{3}\rho$

So we want the EOS for the radiation case. Suppose we have a box uniformly filled with photons, all having the same energy ϵ. Call $\boldsymbol{\pi}$ the momentum of a photon. We assume that the photons are moving in the box isotropically. The magnitude of this vector being $\epsilon = \pi$,

$$\nu = \frac{\#\text{photons}}{V} \,. \tag{12}$$

Hence,

$$\rho = \epsilon\nu \,. \tag{13}$$

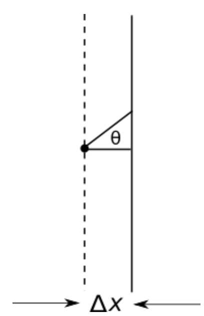

Figure 1. The photon pressure against a ‘wall’ in space.

Regarding Fig. 1, in time interval Δt, how many photons will hit the ‘wall’. In this process, each photon will impart some momentum to the wall by ricocheting off of it head on:

$$\Delta x = c\Delta t \,. \tag{14}$$

At a given angle θ, we get

$$\Delta x = \Delta t \, \cos\theta \,, \quad c = 1 \,, \tag{15}$$

for a photon that will hit the wall. In the x-direction, with momentum π:

$$\Delta\pi = 2\epsilon\cos\theta \,. \tag{16}$$

Hence,

$$\Delta\pi = 2\epsilon\cos\theta\,. \tag{17}$$

Therefore,

$$F_{\substack{\text{unit}\\\text{particle}}} = \frac{\Delta\pi}{\Delta t} = \frac{2\epsilon\cos\theta}{\Delta t}\,. \tag{18}$$

$$F_{\substack{\text{unit}\\\text{particle}}} N = \Delta x A\nu\,. \tag{19}$$

Hence, the total force per unit area

$$P = \frac{2\epsilon\cos x A\nu}{\Delta t}\,. \tag{20}$$

Thus,

$$P(\theta) = 2\epsilon\cos^2\theta\,\nu\,. \tag{21}$$

However, this value has over-estimated by a factor of 2, since photons will travel in both the $+x$ and $-x$ direction, leaving us with the simpler relation

$$P(\theta) = \rho\cos^2\theta\,. \tag{22}$$

Now, let $\mathbf{n}$ be a unit vector with components

$$\mathbf{n} = (n_x, n_y, n_z) = (\cos\theta, n_y, n_z)\,. \tag{23}$$

But, since $\mathbf{n}^2 = 1$ then

$$n_x^2 + n_y^2 + n_z^2 = 1\,. \tag{24}$$

Averaging, we get

$$\langle\, n_x^2 \,\rangle + \langle\, n_y^2 \,\rangle + \langle\, n_z^2 \,\rangle = 1\,. \tag{25}$$

However,

$$\langle\, n_x^2 \,\rangle = \langle\, n_y^2 \,\rangle = \langle\, n_z^2 \,\rangle = \tfrac{1}{3}\,. \tag{26}$$

Therefore,

$$\langle\, \cos^2\theta \,\rangle = \tfrac{1}{3}\,. \tag{27}$$

And finally,

$$P = \tfrac{1}{3}\rho\,. \tag{28}$$

But, where's the 'wall'? Good question. This wall allows as many particles to go through it one way as the other way.

3 Vacuum Energy

Vacuum Energy is an energy assigned to empty space. Thus the energy in a box of fixed volume is a constant. Call the density of this energy ρ_0. Then

$$\rho_0 = \Lambda\frac{3}{8\pi G}\,. \tag{29}$$

Returning to the Friedman equation, we have

$$\begin{aligned}\left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho + \cdots \\ &= \Lambda + \cdots\,. \end{aligned} \tag{30}$$

$$dE = \rho_0 dV = -w\rho_0 dV\,. \tag{31}$$

So, $w = -1$, w refers to a measurement of the vacuum energy, which can be positive or negative. Energy density has the opposite sign to pressure. We don't know how to compute ρ_0, except that it must be small.

Next, let's examine the case of pure vacuum energy. Cases to be distinguished by $\pm$ energy densities

$$\Lambda = \pm, 0\,, \quad k = -1, 0, +1\,, \tag{32}$$

where Λ is proportional to the energy density. So, once again the Friedman equation:

$$\begin{aligned}\left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho_0 - \frac{k}{a^2} \\ &= \Lambda - \frac{k}{a^2}\,.\end{aligned} \tag{33}$$

Next, let's examine the case of pure vacuum energy. Cases to be distinguished by $\pm$ energy densities

$$\Lambda = \pm, 0 \quad k = -1, 0, +1\,. \tag{34}$$

So, once again the Friedman equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \quad (k = -1, 0, 1)\,. \tag{35}$$

Clearly, the case $\Lambda < 0$ and $k > 0$ is nonsense. Let's try an easy case. The 'flat universe' $k = 0$.

$$\left(\frac{\dot{a}}{a}\right)^2 = \Lambda > 0\,. \tag{36}$$

Integrating this, we get

$$a(t) = ce^{\Lambda^{1/2}t} = ce^{Ht}\,, \tag{37}$$

where c is a constant to be determined. H is the Hubble constant $\Lambda^{1/2}$. This space is called *de Sitter space.*

Let's now up the stakes a little by adding another term to the RHS.

$$\left(\frac{\dot{a}}{a}\right)^2 = \Lambda + \frac{c}{a^4}\,. \tag{38}$$

For early times of the universe, a is small and $\frac{1}{a^4}$ dominates over Λ. The current belief is that matter is on a par with Λ for control of the expansion of the universe, with equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \Lambda + \frac{c}{a^3}\,. \tag{39}$$

The universe is progressing from $a(t) \sim t^{2/3}$ to $a(t) \sim e^{Ht}$. Hence

$$\ddot{a}(t) \sim e^{Ht}\,. \tag{40}$$

That is, the acceleration itself is accelerating.

Case $\Lambda > 0$ and $k = +1$:

$$\left(\frac{\dot{a}}{a}\right)^2 = \Lambda - \frac{1}{a^2}\,. \tag{41}$$

Simplifying a bit,

$$\dot{a}^2 - \Lambda a^2 = -1\,. \tag{42}$$

On differentiating by time and then ignoring the case for $\dot{a} = 0$, we get

$$\ddot{a} - \Lambda a = 0\,, \tag{43}$$

with solution $a = \mu \cosh \sqrt{\Lambda} t$, and thus

$$\begin{aligned} \dot{a} &= \mu\sqrt{\Lambda}\sinh\sqrt{\Lambda}t\,, \\ \ddot{a} &= \mu\Lambda\cosh\sqrt{\Lambda}t\,, \end{aligned} \tag{44}$$

This implies that $\mu = \sqrt{\Lambda}$, thus,

$$a = \frac{1}{\sqrt{\Lambda}}\cosh\sqrt{\Lambda}t\,, \tag{45}$$

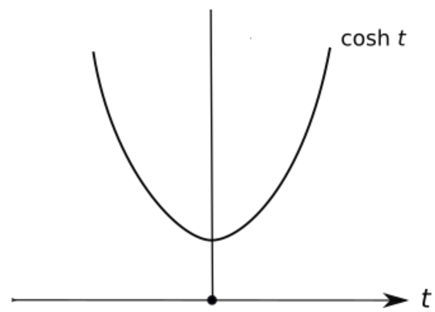

Figure 2. The fast-growing $\cosh(t)$ function for a strange cosmology.

Another picture of de Sitter space

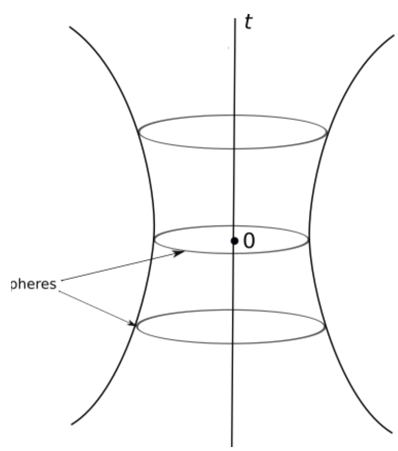

Figure 3. This is also de Sitter space, with positive vacuum energy, $k = +1$.

Now, take $\Lambda < 0$.

$$\left(\frac{\dot{a}}{a}\right)^2 = -|\,\Lambda\,| - \frac{k}{a^2}\,. \tag{46}$$

Now, set $|\,\Lambda\,| = 1$. (Why not?) Then,

$$\left(\frac{\dot{a}}{a}\right)^2 = -1 - \frac{k}{a^2}\,. \tag{47}$$

Obviously, in this case we need to have $k < 0$. If, for example, we take $k = -1$, we get

$$\left(\frac{\dot{a}}{a}\right)^2 = -1 + \frac{1}{a^2}\,, \tag{48}$$

which can be rewritten as

$$\dot{a}^2 + a^2 = 1\,. \tag{49}$$

The graph of this equation is in Fig. 4.

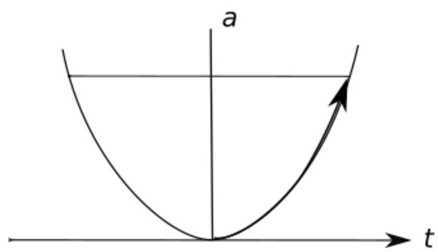

Figure 4. This gives us a harmonic oscillator solution for $a(t)$ with $a \geq 0$.