The Divergence of a Cross Product

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Abstract

Here I prove a formula for the divergence of a cross product, using geometric algebra.

1 Statement of the Problem

Prove the identity

$$\nabla \cdot (\mathbf{E} \times \mathbf{B}) = (\nabla \times \mathbf{E}) \cdot \mathbf{B} - (\nabla \times \mathbf{B}) \cdot \mathbf{E}\,. \tag{1}$$

2 Proof

This proof employs geometric algebra methods.

$$\begin{aligned}
\nabla \cdot (\mathbf{E} \times \mathbf{B}) &= \langle\, \nabla(\mathbf{E} \times \mathbf{B})\,\rangle \\
&= \langle\, \nabla(-i\mathbf{E} \wedge \mathbf{B})\,\rangle \\
&= -i\langle\, \nabla(\mathbf{E} \wedge \mathbf{B})\,\rangle_3 \\
&= -i\langle\, \dot{\nabla}\dot{\mathbf{E}}\dot{\mathbf{B}}\,\rangle_3 \\
&= -i\langle\, \dot{\nabla}\dot{\mathbf{E}}\mathbf{B} + \dot{\nabla}\mathbf{E}\dot{\mathbf{B}}\,\rangle_3 \\
&= -i\langle\, \dot{\nabla}\dot{\mathbf{E}}\mathbf{B}\,\rangle_3 - i\langle\, \dot{\nabla}\mathbf{E}\dot{\mathbf{B}}\,\rangle_3 \\
&= -i\langle\, (\nabla\mathbf{E})\mathbf{B}\,\rangle_3 - i\langle\, \dot{\nabla}\mathbf{E}\dot{\mathbf{B}}\,\rangle_3 \\
&= -i\langle\, (\nabla \wedge \mathbf{E})\mathbf{B}\,\rangle_3 - i\langle\, \dot{\nabla}\mathbf{E}\dot{\mathbf{B}}\,\rangle_3 \\
&= \langle\, (\nabla \times \mathbf{E})\mathbf{B}\,\rangle - i\langle\, \dot{\nabla}\mathbf{E}\dot{\mathbf{B}}\,\rangle_3 \\
&= (\nabla \times \mathbf{E}) \cdot \mathbf{B} - i\langle\, \dot{\nabla}\mathbf{E}\dot{\mathbf{B}}\,\rangle_3\,.
\end{aligned} \tag{2}$$

So far so good. Now to complete the conversion of the second term. If we can interchange the positions of **E** and **B** in the second term, we can pretty much proceed as we did in the first term. Fortunately, we have a standard trick for that. Since

$$2\mathbf{E} \cdot \mathbf{B} = \mathbf{E}\mathbf{B} + \mathbf{B}\mathbf{E}\,, \tag{3}$$

then

$$\mathbf{EB} = 2\mathbf{E}\cdot\mathbf{B} - \mathbf{BE}\,. \tag{4}$$

Therefore,

$$\begin{aligned} -i\langle\, \dot{\nabla}\mathbf{E}\dot{\mathbf{B}}\,\rangle_3 &= -i\langle\, \dot{\nabla}(2\mathbf{E}\cdot\dot{\mathbf{B}} - \dot{\mathbf{B}}\mathbf{E})\,\rangle_3 \\ &= -i\langle\, -\dot{\nabla}\dot{\mathbf{B}}\mathbf{E}\,\rangle_3\,, \end{aligned} \tag{5}$$

since $\dot{\nabla}(2\mathbf{E}\cdot\dot{\mathbf{B}})$ has no pseudoscalar part. Now, following nearly the same process we used before,

$$-i\langle\, -\dot{\nabla}\dot{\mathbf{B}}\mathbf{E}\,\rangle_3 = -(\nabla\times\mathbf{B})\cdot\mathbf{E}\,. \tag{6}$$

Now, substituting the result we got from this last equation into (2), we get

$$\nabla\cdot(\mathbf{E}\times\mathbf{B}) = (\nabla\times\mathbf{E})\cdot\mathbf{B} - (\nabla\times\mathbf{B})\cdot\mathbf{E}\,. \tag{7}$$

Done.