Fixing Ampere's Law

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Abstract

Here we modify Ampere's Law to be consistent with the equation of continuity. By the way, this is old stuff.

1 Statement of the Problem

Ampere's Law can be stated in its differential form as:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}\,, \tag{1}$$

where $\mathbf{B}$ is the magnetic field, $\mathbf{J}$ is the electric current density, and μ_0 is the permeability of free space. The *equation of continuity* for charges is

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0\,, \tag{2}$$

where ρ is the charge density.

Now, what would happen if we took the divergence of both sides of (1)? Given that the divergence of a curl is zero, we would get

$$0 = \mu_0 \nabla \cdot \mathbf{J} = -\mu_0 \frac{\partial \rho}{\partial t}\,. \tag{3}$$

But $\partial\rho/\partial t$ is not identically zero, so we had better fix this! But how? How about introducing our old friend $\mathbf{x}$, where it's an unknown vector quantity at this point. Let's assume that the fix is as simple as modifying (1) by adding $\mathbf{x}$ to it and then solving for this vector.[1]

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mathbf{x}\,. \tag{4}$$

So now when we take the divergence of both sides of (4), we get

$$\nabla \cdot \mathbf{x} = \mu_0 \frac{\partial \rho}{\partial t}\,. \tag{5}$$

[1] I don't know where I got the idea for this 'fix'. It's probably quite old.

Now we need one more equation: Gauss's Law for the electric field.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}\,. \tag{6}$$

On differentiating this through by t, we get

$$\nabla \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\epsilon_0}\frac{\partial \rho}{\partial t}\,. \tag{7}$$

Combining this last equation and (5), we get that

$$\nabla \cdot \mathbf{x} - \epsilon_0\mu_0 \nabla \cdot \mathbf{E} = 0\,. \tag{8}$$

So, a possible solution to $\mathbf{x}$ is[2]

$$\mathbf{x} = \epsilon_0\mu_0 \frac{\partial \mathbf{E}}{\partial t}\,. \tag{9}$$

Thus, Ampere's Law becomes

$$\nabla \times \mathbf{B} = \mu_0\mathbf{J} + \epsilon_0\mu_0\frac{\partial \mathbf{E}}{\partial t} = \mu_0\big(\mathbf{J} + \epsilon_0\frac{\partial \mathbf{E}}{\partial t}\big)\,, \tag{10}$$

which is known as the Ampere-Maxwell Law.

[2]As is common to finding the solution to a differential equation, the solution may not be unique.