

The Time Rate of Change of the EMF Energy

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Abstract

Here's something interesting to do with Maxwell's equations: Calculate the time rate of change of the electromagnetic field energy. It's also known as Poynting's Theorem.

1 Statement of the Problem

Prove the identity

$$\frac{\partial U}{\partial t} = -\nabla \cdot \mathbf{S} - \mathbf{J} \cdot \mathbf{E}, \quad (1)$$

where \mathbf{B} is the magnetic field, \mathbf{J} is the electric current density, and U is the electromagnetic field energy, given by

$$U = \frac{1}{2} \frac{1}{\mu_0} \mathbf{B}^2 + \frac{1}{2} \epsilon_0 \mathbf{E}^2, \quad (2)$$

where μ_0 is the permeability of free space, and ϵ_0 is the permittivity of free space. We also have that \mathbf{S} is the Poynting vector, defined as

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}. \quad (3)$$

2 Lemma

The following identity is the so-called *Product Identity*, which entails the divergence of a cross product:

Let \mathbf{A} and \mathbf{B} be arbitrary vectors in Euclidean 3-space. Then

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - (\nabla \times \mathbf{B}) \cdot \mathbf{A}. \quad (4)$$

The proof to this lemma is given in another paper in this same section. The equation is fundamental to the proof of Poynting's Theorem.

3 Proof

This proof is merely the manipulation of vector algebra/calculus equations. We begin rewriting (4) in terms of our E&M variables, getting

$$\nabla \cdot (\mathbf{E} \times \mathbf{B}) = (\nabla \times \mathbf{E}) \cdot \mathbf{B} - (\nabla \times \mathbf{B}) \cdot \mathbf{E}. \quad (5)$$

Now, from Faraday's Law, we have that

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \quad (6)$$

and from the Ampere-Maxwell Law, we have that

$$\nabla \times \mathbf{B} = \mu_0 (\mathbf{J} + \epsilon_0 \partial_t \mathbf{E}). \quad (7)$$

So, on substituting these last two equations and Eq. (3) into Eq. (5), we get

$$\mu_0 \nabla \cdot \mathbf{S} = (-\partial_t \mathbf{B}) \cdot \mathbf{B} - \mu_0 (\mathbf{J} + \epsilon_0 \partial_t \mathbf{E}) \cdot \mathbf{E}, \quad (8)$$

which can be written as

$$\nabla \cdot \mathbf{S} = -\left(\frac{1}{2} \frac{1}{\mu_0} \partial_t \mathbf{B}^2 + \frac{1}{2} \epsilon_0 \partial_t \mathbf{E}^2\right) - \mathbf{J} \cdot \mathbf{E}, \quad (9)$$

Employing (2), we get

$$\nabla \cdot \mathbf{S} = -\frac{\partial U}{\partial t} - \mathbf{J} \cdot \mathbf{E}, \quad (10)$$

and this is equivalent to (1).