An Amplification of Oblique Collisions from *Analytical Mechanics*, Chapter 7

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1 Introduction

Here we present an amplification of oblique collisions from the physics textbook *Analytical Mechanics*, Chapter 7, Grant R. Fowles (Holt, Rinehart, and Winston), New York, 1962, p. 148–9. I'll be using the basic notation that Fowles used, but I'll add to it a bit more explanation.

By oblique collisions we mean collisions that the colliding particles after the collision are not scattered off each other along the original line of impact; hence, we need to consider their scattering angles relative to the impact line in some scattering plane. In Figure 1 we see the basic situation in the laboratory frame.

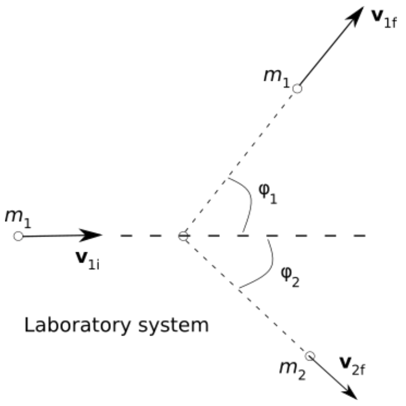

Figure 1. In the lab frame, the stationary target of mass m_2 is approached by the moving projectile (incident) particle of mass m_1. The projectile particle scatters 'up' at an angle φ_1 to the line of impact. The target particle scatters 'down' at angle φ_2. We will not assume the collision to be elastic, so we'll have to solve for the required results using momentum conservation.

Momentum conservation in the lab frame requires that

$$m_1\mathbf{v}_{1i} = m_1\mathbf{v}_{1f} + m_2\mathbf{v}_{2f}\,. \tag{1}$$

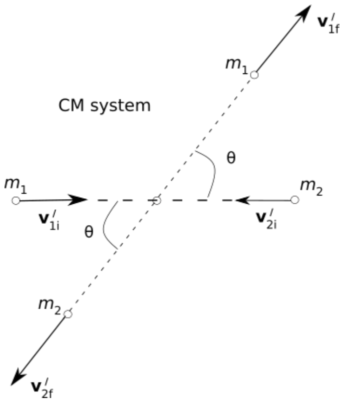

Figure 2. In the CM frame, the momenta of the two particles are equal and opposite to each other. Thus, in this frame, they cancel each other out. As a consequent, this cancelling out of momentum survives the collision.

2 The Solution

By construction, in the center-of-mass (CM) frame, momentum adds up to zero, hence

$$m_1\mathbf{v}'_{1i} + m_2\mathbf{v}'_{2i} = \mathbf{0}\,. \tag{2}$$

But since our system of particles is not subject to external forces, the momentum after collision is equal to the momentum before collision, hence:

$$m_1\mathbf{v}'_{1f} + m_2\mathbf{v}'_{2f} = \mathbf{0}\,. \tag{3}$$

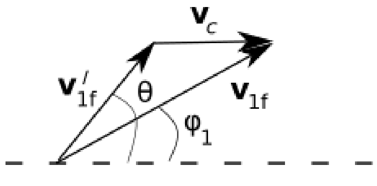

Figure 3. When we view the interaction from the CM frame, the speed of the incident particle in the lab frame is moving slower relative to this frame by the amount v_c, where this is the speed of the CM frame as seen by the lab frame. Thus we have to subtract off from all lab-frame speeds along the direction of the impact line, keeping the speeds of objects perpendicular to this line the same.

Our goal is to determine the scattering angles φ_1 and φ_2 as presented in Fig. 1. To accomplish this, we'll transform into the CM frame and then return to the Lab Frame. A great deal of our solution will be geometric, or more specifically, trigonometric.

Now, if we can get φ_1 as a function of θ, we are more than halfway finished. First, let's redo the last figure, adding in some convenient parameters.

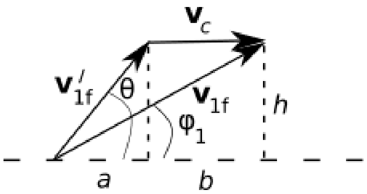

Figure 4. For easier computation, I added labels a, b to the horizontal line segments, and h to the vertical line segment.

From Fig. 4, we derive the following relations:

$$a + b = v_{1f} \cos \varphi_1 \,, \tag{4a}$$

$$a = v'_{1f} \cos \theta \,, \tag{4b}$$

$$b = v_c \,, \tag{4c}$$

$$h = v'_{1f} \sin \theta = v_{1f} \sin \varphi_1 \,. \tag{4d}$$

On combining the first three of these, we get

$$v_{1f} \cos \varphi_1 = a + b = v'_{1f} \cos \theta + v_c \,. \tag{5}$$

We also get

$$\frac{v_{1f} \sin \varphi_1}{v_{1f} \cos \varphi_1} = \frac{v'_{1f} \sin \theta}{a + b} \,, \tag{6}$$

or

$$\begin{aligned} \tan \varphi_1 &= \frac{v'_{1f} \sin \theta}{v'_{1f} \cos \theta + v_c} \\ &= \frac{\sin \theta}{\cos \theta + v_c / v'_{1f}} \\ &= \frac{\sin \theta}{\cos \theta + \alpha} \,, \end{aligned} \tag{7}$$

where $\alpha \equiv v_c / v'_{1f}$.

Now, from the definition of the CM frame, we have that

$$(m_1 + m_2)\mathbf{v}_c = m_1 \mathbf{v}_{1i} + \cancelto{0}{m_2 \mathbf{v}_{2i}} . \tag{8}$$

Therefore,

$$\mathbf{v}_c = \frac{m_1}{m_1 + m_2}\mathbf{v}_{1i}\,. \tag{9}$$

We can extract the scalar values of this last equation to get

$$v_c = \frac{m_1}{m_1 + m_2} v_{1i}\,. \tag{10}$$

Hence, α becomes

$$\alpha = \frac{v_c}{v'_{1f}} = \frac{m_1 v_{1i}}{(m_1 + m_2)v'_{1f}}\,. \tag{11}$$

Now, we have the conservation of momentum equations in scalar form from (2) and (5):

$$m_1 v'_{1i} - m_2 v'_{2i} = 0\,, \tag{12}$$

and

$$m_1 v'_{1f} - m_2 v'_{2f} = 0\,. \tag{13}$$

Using (13), we can further refine (11) to

$$\alpha = \frac{m_1 v_{1i}}{(m_1 + m_2)v'_{1f}} = \frac{m_1 v_{1i}}{m_2(v'_{1f} + v'_{2f})}\,. \tag{14}$$

Observing again the CM view of the collision in Fig. 2, we see that the quantity $v'_{1f} + v'_{2f}$ represents the relative speed of the two particles (as seen in that frame) after collision, which we'll designate as v_r; hence,

$$\alpha = \frac{m_1 v_{1i}}{m_2 v_r}\,. \tag{15}$$

Now we review our final equation under some special situations. The first being of an elastic collision. The effect of this is to set $v_r = v_{1i}$.[1]

Now we turn to special cases. First up is the question of what happens when $m_1 = m_2$? We'll answer this question under the assumption of an elastic collision. With both of these assumptions, we have that $\alpha = 1$. From (7) we get

$$\tan\varphi_1 = \frac{\sin\theta}{\cos\theta + 1}\,. \tag{16}$$

Using a trig identity, we get

$$\tan\varphi_1 = \tan\,\frac{\theta}{2}\,. \tag{17}$$

Hence, $\varphi_1 = \theta/2$.

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[1] Establishing this equality may be easy, but I didn't find that so.

Problem: Show that, under elastic, oblique collision of two particles of equal mass, the sum of the scattering angles is $\pi/2$.

We can begin by dropping the masses from (1) and separating that result into components:

$$\begin{aligned} v_{1i} &= v_{1f}\cos\varphi_1 + v_{2f}\cos\varphi_2\,, & \text{(18a)}\\ 0 &= v_{1f}\sin\varphi_1 - v_{2f}\sin\varphi_2\,. & \text{(18b)} \end{aligned}$$

On squaring these equations, we get

$$\begin{aligned} v_{1i}^2 &= v_{1f}^2\cos^2\varphi_1 + 2v_{1f}v_{2f}\cos\varphi_1\cos\varphi_2 + v_{2f}^2\cos^2\varphi_2\,, & \text{(19a)}\\ 0 &= v_{1f}^2\sin^2\varphi_1 - 2v_{1f}v_{2f}\sin\varphi_1\sin\varphi_2 + v_{2f}^2\sin^2\varphi_2\,. & \text{(19b)} \end{aligned}$$

Adding these together, we have that

$$\begin{aligned} v_{1i}^2 &= v_{1f}^2 + 2v_{1f}v_{2f}(\cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2) + v_{2f}^2\\ &= v_{1f}^2 + 2v_{1f}v_{2f}\cos(\varphi_1+\varphi_2) + v_{2f}^2\,. & \text{(20)} \end{aligned}$$

We haven't yet used the conservation of kinetic energy in this elastic collision. Thus,

$$v_{1i}^2 = v_{1f}^2 + v_{2f}^2\,. \tag{21}$$

On comparing these last two equations, we are forced to set $\cos(\varphi_1+\varphi_2) = 0$. And that, in turn, forces us to conclude that $\varphi_1+\varphi_2 = \pi/2$.