A Neat Physics Problem

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February 19, 2020

Abstract

This paper is a redo of an article that first appeared in the *Arizona Journal of Natural Philosophy*, April, 1991. This problem demonstrates a number of useful heuristics in mathematics and physics.

This problem is from *University Physics* (Sears & Zemansky, Addison Wesley 4th ed), page 421: 29-10. It asks the student to prove that the resistance of the following network is equal to $(1 + \sqrt{3})r$.

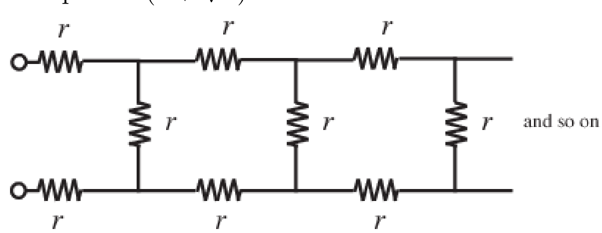


Figure 1

Although on retrospect, I have a shorter solution to present, I'll reproduce the original, longer version of the proof because it still has many useful heuristic tricks to present.

1 Problem and Solution

The first step in solving this problem is to solve a simpler related problem. Let's start with the one in Figure 2. Then I'll redraw it as is shown in Figure 3.

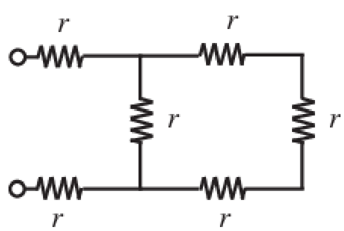


Figure 2

The resistance of the resistance combination in Figure 3 is $2r + \dfrac{r(3r)}{r + 3r}$.

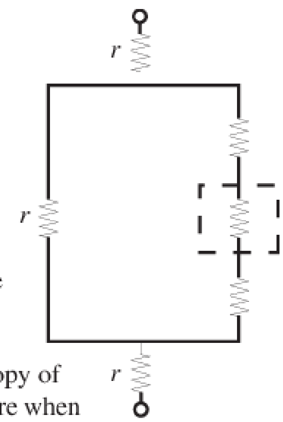


Figure 3.

Redrawing Figure 2 helps the problem solver to see the right approach. In the dotted area the resistor will be replaced by a copy of the present figure when the next loop is added.

By adding another loop as in Figure 4, we might hope to find a pattern.

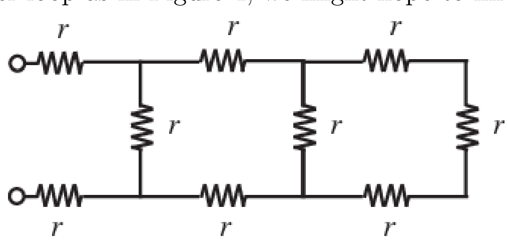
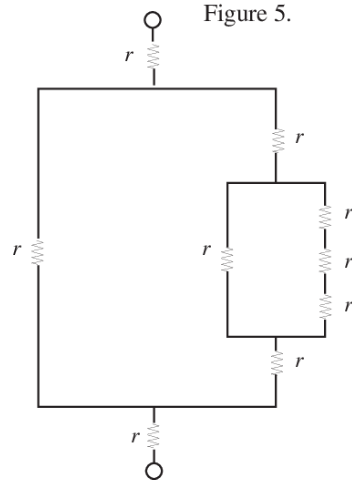


Figure 4

On redrawing the figure we get Figure 5:

Now we can look for some pattern that is operating. Comparing Figure 5 with Figure 3, we can see that by adding a loop we have replaced the outlined part of Figure 3 with a copy of all of Figure 3.

Figure 5.



So now we have a recursive definition of the resistance. The resistance of the setup with $i+1$ loops, R_{i+1}, is a function of the resistance, R_i, of the setup with i loops, where we imagine adding the next loop in the succession on the left.

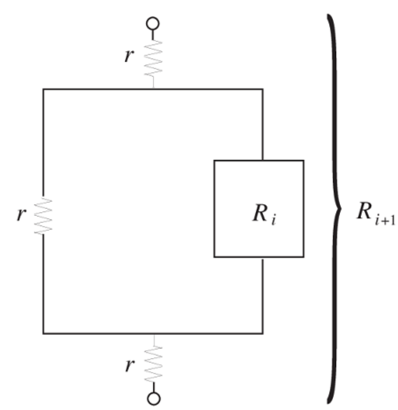


Figure 6

We're now left with a rather simple resistance network by which we can easily compute thre resistance R_{i+1}.

$$R_{i+1} = 2r + \frac{rR_i}{r+R_i}\,. \tag{1}$$

In the limit $i \to \infty$, $R_{i+1} = R_i \equiv R$ therefore,

$$R = 2r + \frac{rR}{r+R}\,, \tag{2}$$

which is a quadratic with roots $(1 \pm \sqrt{3})r$.

Thus the "physical" root is $(1 + \sqrt{3})r$. In fact, this root cannot literally be interpreted as "physical," as the given network cannot be physically attained.

When I first published this article, I got some negative feedback from people who thought that I had no right to claim that the answer to the problem is "nonphysical." To me, the semantic issue is whether or not we can claim that an answer is 'physical' is it corresponds to an unphysical situation, which clearly an infinite network must be.

2 Afterthoughts: A Teachable Moment

Though my solution is correct, there is a quicker solution, which is based on the notion of self-similarity. Now, I'm just an amateur in math and physics, but one thing I have learned over the decades of my problem solving in mathematics: the broader the base of one's knowledge in mathematics, the generally better one's problem-solving skills will be.

Case in point #1: In the early 1990s, I made an accidental discovery of the Chebyshev polynomials sitting within the unipodal numbers I was studying at the time. However, at that time, I knew nothing of those polynomials, so I didn't recognize them for what they were and missed out on an opportunity to investigate them at that time. It was only after I had made a broader study of mathematics that I realized in 2019 what I had discovered a quarter-century earlier, and then wrote up two papers on them.

Case in point #2: My recent study in continued fractions, which seemed unrelated to anything in the area of study I'm mainly interested in (mathematical physics), I made this interesting discovery (though not original to me, of course). Let me demonstrate.

What's the value of x defined in the continued fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}}}}} ? \tag{3}$$

Here's where self-similarity comes in. Notice that the expression in the first-level denominator is itself equal to x. Thus, this last equation becomes

$$x = 1 + \frac{1}{x}\,, \tag{4}$$

which has solution

$$x = \frac{1+\sqrt{5}}{2}\,, \tag{5}$$

which, by the way, is the Golden Ratio!

The take-away point here is that we should see an immediate analogy between the structure of the continued fraction in Eq. (3) and in the circuit diagram in Figure 1. I now assume that the authors of the problem had desired that the student would see the self-similarity aspect of the circuit and imagine the subnetwork as shown below in Figure 7, as the portion of the full network to the right of line segment AB, is self-similar to the entire network:

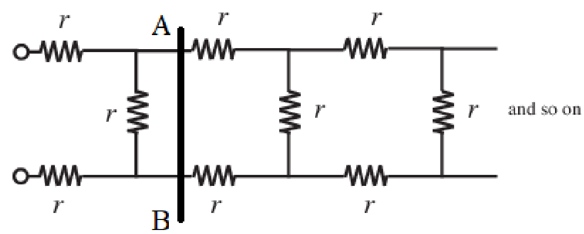


Figure 7

Hence, we would be justified to draw the 'equivalent' circuit diagram in Figure 8, justifying an immediate writing down of Eq. (2).

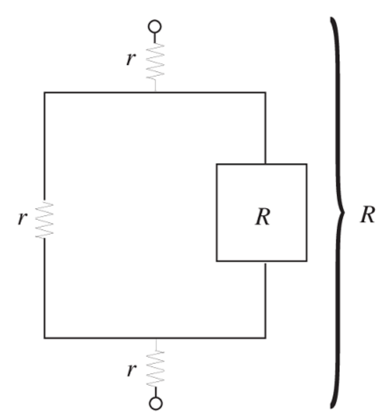


Figure 8

So, why didn't I think of using self-similarity when I first encountered this problem? I think it's simply because we are trained to take cases involving infinity and reduce them to a finite case, indexed by some indexing variable, and then let the index go to infinity.

Two points left to make: First, it's not a good idea to become too narrowly focused in one's study of mathematics. Second, one may find, in the process of time, that a field of mathematics that seems quite unrelated to some other field of study, may turn out to have a profound connection to it.

What's the difference between knowledge and wisdom? Knowledge is to be aware of all your options. Wisdom is to know which is the best one to choose in a given circumstance.