The Frenet-Serret Equations

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February 17, 2020

Abstract

This paper is a redo of an article that first appeared in the *Arizona Journal of Natural Philosophy*, July, 1996.The Frenet-Serret equations were first developed independently by both Frenet and Serret in the nineteenth century. The equations are a system of three first-order vector differential equations for the vectors of a triad of orthonormal vectors along a smooth space curve, and we shall be solving for these differential equations. Most of what follows requires only a knowledge of standard calculus, though some small sections require a knowledge of geometric algebra.

Consider the parameterized curve $\mathcal{C}$ in Euclidean 3-space, $\mathbb{R} \to E^3$, $\mathbf{x} = \mathbf{x}(t)$, where t is a continuous scalar variable. We assume that $\mathcal{C}$ is differentiable to all orders. We can define the velocity $\mathbf{v}$ of the curve at $\mathbf{x}(t)$ as

$$\mathbf{v} \equiv \dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt}\,, \tag{1}$$

and the speed of the curve at the same point as

$$v = |\,\mathbf{v}\,|\,. \tag{2}$$

If v is a known function of t we can set up another parameterization of the curve:

$$s \equiv \int_{\mathbf{x}_0}^{\mathbf{x}(t)} |\,d\mathbf{x}\,| = \int_{t_0}^{t} \left|\,\frac{d\mathbf{x}}{dt}\,\right| dt = \int_{t_0}^{t} v\,dt\,. \tag{3}$$

By a basic calculus theorem we know that $ds/dt = v$. The velocity vector can be written in the form

$$\mathbf{v} = v\hat{\mathbf{v}} = v\mathbf{T}\,, \tag{4}$$

where $\mathbf{T} = \hat{\mathbf{v}}$ and $\hat{\mathbf{v}}^2 = \mathbf{T}^2 = 1$. The unit vector $\mathbf{T}$ is a unit tangent vector to the curve at the point $\mathbf{x}(t)$, but we will want to derive $\mathbf{T}$ as a function of s, for then the ensuing equations are in a simpler form.

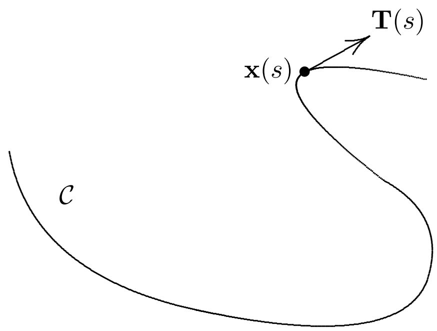


Figure 1. On our smooth parameterized curve $\mathcal{C}$ we can put the unit tangent vector $\mathbf{T}(s)$ at point $\mathbf{x}(s)$.

Thus,

$$\mathbf{T} = \frac{\mathbf{v}}{v} = \frac{d\mathbf{x}/dt}{ds/dt} = \frac{d\mathbf{x}}{ds}\,. \tag{5}$$

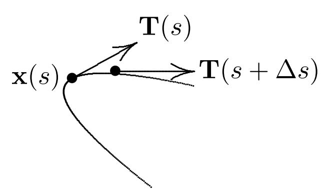
(See Fig. 1.)

We introduce the notation of a dot over a variable as meaning differentiation by the variable t. Since $\mathbf{T}^2 = 1$ then $\mathbf{T}\cdot\dot{\mathbf{T}} = 0$. Now we introduce a vector $\mathbf{N}$ such that

$$\mathbf{N} \equiv \frac{\dot{\mathbf{T}}}{|\,\dot{\mathbf{T}}\,|} = \frac{d\mathbf{T}/ds}{|\,d\mathbf{T}/ds\,|}\,. \tag{6}$$

The vector $\mathbf{N}$ is clearly a unit vector perpendicular to $\mathbf{T}$. It is called the *principal normal* to the curve at $\mathbf{x}(s)$.

Figure 2a. The relative angle between $\mathbf{T}(s)$ at point $\mathbf{x}(s)$ and $\mathbf{T}(s+\Delta s)$ at point $\mathbf{x}(s+\Delta s)$ will be denoted by θ.



At this point we need to introduce the limit of a trigonometric function to sire an important characteristic of curves, namely *curvature* at a point. The *curvature* κ of the curve $\mathcal{C}$ at the point $\mathbf{x}(s)$ is defined to be the limit of the change in angle between two tangents to the curve, $\mathbf{T} = \mathbf{T}(s)$ and $\mathbf{T}_1 = \mathbf{T}(s+\Delta s)$ (see Fig 2a), as $\Delta s \to 0$, or

$$\kappa \equiv \lim_{\Delta s\to 0} \frac{\theta}{\Delta s}\,, \tag{7}$$

where θ is the angle between the two tangents (see Fig. 2b). From Fig. 2b we get the trigonometric relation $\sin\frac{1}{2}\theta = \frac{1}{2}|\,\Delta\mathbf{T}\,|$. But as $\theta \to 0$ and $\Delta s \to 0$, then

$\sin \frac{1}{2}\theta$ goes as just $\frac{1}{2}\theta$.

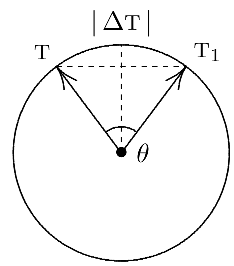


Figure 2b. The relative angle θ between $\mathbf{T} = \mathbf{T}(s)$ and $\mathbf{T}_1 = \mathrm{T}(s + \Delta s)$ is clear in this depiction in which the tails are joined at the origin of a unit circle. Note that the horizontal dashed line is given by $|\,\Delta\mathbf{T}\,| = |\,\mathbf{T}_1 - \mathbf{T}\,|$.

Thus

$$\kappa = \lim_{\Delta s \to 0} \frac{|\,\Delta\mathbf{T}\,|}{\Delta s} = \left|\,\frac{d\mathbf{T}}{ds}\,\right| . \tag{8}$$

So, from Equation (6) we can write

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}\,, \tag{9}$$

which constitutes our first differential equation of the desired triad. Furthermore, we can define the *radius of curvature at the point* $\mathbf{x}(s)$ as

$$\rho \equiv \frac{1}{\kappa}\,. \tag{10}$$

The vector $\mathbf{N}$ is depicted in Fig. 3.

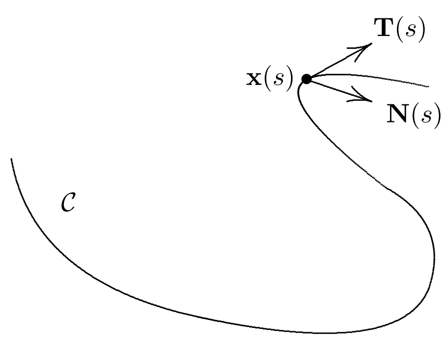


Figure 3. On our smooth parameterized curve $\mathcal{C}$ we can put the unit tangent vector $\mathbf{T}(s)$ at point $\mathbf{x}(s)$ and the principal (unit) normal vector $\mathbf{N}(s)$.

Before we continue our search for the other two differential equations of our triad, we pause to consider the notion of acceleration along our space curve. We can define acceleration **a** as

$$\begin{aligned}\mathbf{a} \equiv \dot{\mathbf{v}} &= \frac{d}{dt}\,(v\mathbf{T}) \\ &= \dot{v}\mathbf{T} + v\dot{\mathbf{T}} \\ &= \dot{v}\mathbf{T} + v\,\frac{ds}{dt}\,\frac{d\mathbf{T}}{ds} \\ &= \dot{v}\mathbf{T} + \frac{v^2}{\rho}\,\mathbf{N}\,. \end{aligned} \tag{11}$$

Or we can write

$$\mathbf{a} = a_t\mathbf{T} + a_n\mathbf{N}\,,$$

where $a_t = \dot{v}$ and $a_n = v^2/\rho$.

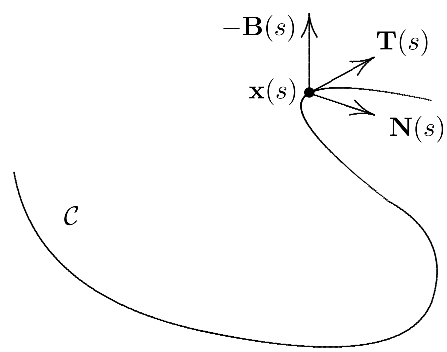


Figure 4. On our smooth parameterized curve $\mathcal{C}$ we can put the unit tangent vector $\mathbf{T}(s)$, unit principal normal vector $\mathbf{N}(s)$, and unit binormal vector $\mathbf{B}(s) \equiv \mathbf{T}\times\mathbf{N}$ at point $\mathbf{x}(s)$. We have depicted $-\mathbf{B}$ rather than $\mathbf{B}$ for improved visualization.

So far we have two orthonormal vectors of our triad, $\mathbf{T}$ and $\mathbf{N}$, and only one differential equation, that for $d\mathbf{T}/ds$. We can complete our set of unit basis vectors in our triad by defining the last as

$$\mathbf{B} \equiv \mathbf{T}\times\mathbf{N}\,, \tag{12}$$

where $\mathbf{B}$ is said to be the *binormal* vector. The three vectors $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$ form a righthanded, orthonormal basis at a given point $\mathbf{x}(s)$ on the curve. (See Fig. 4.) At any such point we can form a unit pseudoscalar $i = \mathbf{T}\wedge\mathbf{N}\wedge\mathbf{B}$. Now, one can by ordinary methods show that $\mathbf{B}$ is indeed a unit vector, but I wish to demonstrate how to do it in geometric algebra. First, we write

$$\mathbf{B} = \mathbf{T}\times\mathbf{N} \equiv -i\mathbf{T}\wedge\mathbf{N} = -i\mathbf{T}\mathbf{N}\,, \tag{13}$$

where, of course, $\mathbf{T}\cdot\mathbf{N} \equiv 0$. Now

$$\mathbf{B}^2 = \mathbf{B}\mathbf{B}^\dagger = (-i\mathbf{T}\mathbf{N})(i\mathbf{N}\mathbf{T}) = (-i^2)\mathbf{T}(\mathbf{N}\mathbf{N})\mathbf{T} = \mathbf{T}\mathbf{T} = 1\,. \tag{14}$$

Thus $\mathbf{B}$ is a unit vector.

Now, although we introduced the vector $\mathbf{N}$ before the vector $\mathbf{B}$, we shall find it easier to find $d\mathbf{B}/ds$ before $d\mathbf{N}/ds$. So, from (14) we can conclude that

$$\frac{d\mathbf{B}}{ds} \cdot \mathbf{B} = 0\,, \tag{15}$$

therefore

$$\frac{d\mathbf{B}}{ds} = \alpha\mathbf{T} + \beta\mathbf{N}\,, \tag{16}$$

where α and β are scalar coefficients to be determined. First, we show that $\alpha = 0$ by showing that $d\mathbf{B}/ds$ is orthogonal to $\mathbf{T}$. By differentiating $\mathbf{B}$ in (13) we get

$$\frac{d\mathbf{B}}{ds} = \frac{d\mathbf{T}}{ds} \times \mathbf{N} + \mathbf{T} \times \frac{d\mathbf{N}}{ds} = \mathbf{T} \times \frac{d\mathbf{N}}{ds}\,, \tag{17}$$

where the first term is zero because $d\mathbf{T}/ds$ and $\mathbf{N}$ are scalars multiples of each other. What (17) tells us is that $d\mathbf{B}/ds$ is orthogonal to $\mathbf{T}$ (why?). Thus,

$$\frac{d\mathbf{B}}{ds} = \beta\mathbf{N}\,. \tag{18}$$

We will follow standard procedure at this point and replace our β by $-\tau$. Thus,

$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}\,, \tag{19}$$

where τ is referred to as the *torsion* to the curve at point $\mathbf{x}(s)$.

Now, $d\mathbf{B}/ds$ is in either the positive or negative $\mathbf{N}$ direction; either way we can interpret this as meaning that the two vectors $\mathbf{B}$ and $\mathbf{N}$ are "rotating" around the $\mathbf{T}$ vector. And we can think of this effect as a "twisting or torsion of the curve itself." The reason to introduce τ and the minus sign is for a simple convention, that is, to ensure that, as s increases, the torsion is positive when the $\mathbf{B}$ and $\mathbf{N}$ are rotating around the $\mathbf{T}$ vector in the same way as a righthanded screw would as it advances.

There is a peculiar difference of agreement in the literature on whether the torsion is only nonnegative or else is any real value. It is claimed here to be generally any real value, and only otherwise restricted by the particular curve under investigation.

Thus we now have two of our the three desired differential equations. We can get our third, and last, equation by differentiating

$$\mathbf{N} = \mathbf{B} \times \mathbf{T} \tag{20}$$

to get

$$\begin{aligned}\frac{d\mathbf{N}}{ds} &= \frac{d\mathbf{B}}{ds} \times \mathbf{T} + \mathbf{B} \times \frac{d\mathbf{T}}{ds} \\ &= -\tau\mathbf{N} \times \mathbf{T} + \mathbf{B} \times (\kappa\mathbf{N}) \\ &= (-\tau)(-\mathbf{B}) + \kappa(-\mathbf{T}) \\ &= \tau\mathbf{B} - \kappa\mathbf{T}\,. \end{aligned} \tag{21}$$

Thus the Frenet-Serret differential equations for the system of righthanded orthornormal unit vectors along the space curve $\mathcal{C}$ at any point s are given by

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}\,,$$
$$\frac{d\mathbf{N}}{ds} = \tau\mathbf{B} - \kappa\mathbf{T}\,,$$
$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}\,. \tag{22}$$