The Rocket Equation – It's a Blast

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Abstract

Paradoxically, we conceptualize the variable-mass rocket as a system of two invariable-mass subsystems.

1 Introduction

The variable-mass rocket problem finds its way into first-year physics with calculus. Solving the differential equation that results from the physical analysis of the problem is easy. However, finding that equation can be hard for the student, first because the method of obtaining that equation is often left poorly described, and, second, because there are many ways to go about modeling the situation prior to applying Newton's equations. It all boils down to how we divide the rocket (the 'system') into two subsystems that can be easily analyzed in the change-of-momentum expression.

In the Newtonian game of physical analysis, the system is influenced by its environment through forces the environment makes on the system. If the system is undergoing accelerated motion, we'll need to find the appropriate equations of motion and then solve them for whatever information we desire, such as velocity or position as a function of time.

We must reference positions and velocities from some arbitrarily chosen inertial reference frame. In practice, this frame will likely be only approximately inertial, which will certainly be the case if we choose a frame attached to the surface of the earth, since the earth's motion is non-inertial.

2 Statement of the Problem

We are asked to find the velocity V of a rocket launched from the surface of the earth as a function of time t, by 1) ignoring variations in the earth's gravitational field as a function of height, 2) ignoring all atmospheric considerations, and 3) treating the launch pad as an approximate inertial reference frame, for the duration (until its fuel is spent). Furthermore, 4) we assume that the speed of the ejected exhaust (consumed fuel) is a constant relative to the rocket, and we'll call it ν. (I assume that the constancy at which the exhaust leaves the rocket is the result of precise engineering.) Lastly, 5), we're assuming that the rocket moves only vertically relative to the launch platform.

It's easy to miss that we have implied the theoretical existence of a comoving inertial frame, which we can imagine to be 'attached' to the rocket, by which we can meaningfully assign velocities to the exhaust lump of mass Δm. The particular 'comoving rest frame' changes instant to instant, because the rocket is assumed to continuously accelerate so long as it has fuel left to consume. And, the speed of the instantaneous inertial reference frame relative to the ground-fixed frame is the same as the speed of the rocket itself at time t.

We assume that the rate in time at which the fuel is consumed is a constant, which we'll call κ. Thus, if we use M for the mass of the rocket system, i.e., the mass of the rocket and its fuel at time t, then

$$\kappa = -\frac{dM}{dt}\,, \tag{1}$$

the minus sign appearing because the mass is decreasing in time and we wish to make κ a positive number.

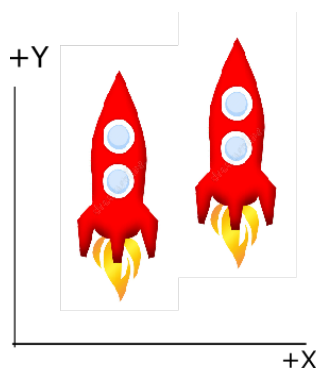

Figure 1. The rocket on the left represents the system at time t, and the rocket on the right represents the system at time $t + dt$. The question is, How ought we to define the 'system' so that the impulse-momentum equation makes sense when applied to it? The reference frame is fixed to the launch pad.

Our goal now is to obtain a differential equation of motion in time to solve for the velocity $V = V(t)$. So, according to how we've setup the problem, the only axis of interest will be the Y-axis (vertical axis), along which the motion is constrained to exist in time. Hence, velocities in the direction of +Y are positive, and those in the opposite direction are negative. See Figure 1.

3 Modeling the rocket and its fuel

Let's begin by defining our ground-based inertial reference frame. We take the launch pad as the origin of coordinates, treating the earth as though it were fixed with respect to the distant stars as a first approximation.

This matter of obtaining a differential equation of velocity in time will follow the standard approach used in physics problems: First, we identify the state of the system at some arbitrary time t, and then determine how the system evolves under the influence of whatever affects it, according to the equations given us by Newton or by equations extrapolated by them. This we will attempt to do by using the approximate Impulse-Momentum relation

$$\sum F_{\mathrm{sys}} \Delta t = \Delta p_{\mathrm{sys}} \,, \tag{2a}$$

which stresses finite, rather than infinitesimal differences. In words, we say that the sum of the impulses on the system (of duration Δt) is (approximately) equal to the change in momentum of the system, and later take the limit as $\Delta t \to dt$ to arrive at exactness[1]:

$$\sum F_y dt = dp_{\mathrm{sys}} \,. \tag{2b}$$

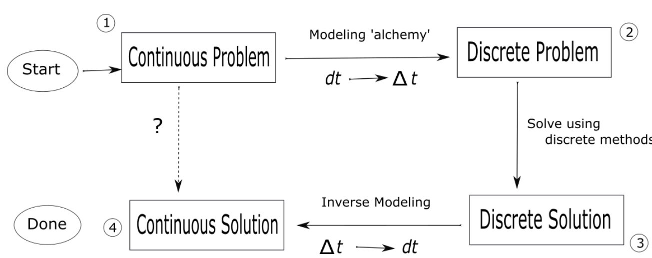

Figure 2. Here we present a graphical representation of the Big Picture overview of how we will solve the first part of this problem, which is to find the differential equation for V. What we have represented in the figure is our attempt to map the conceptual model of continuous motion and forces into an easier conceptual framework of discrete models and discrete methods.

In words, we have that the impulse to a system (from all external forces acting on it) is equal to the change of momentum of the system, which, in turn, is equal to the sum of the changes of momenta of all the subsystems of the system. By our arbitrary design, we have two subsystems: I) the rocket and fuel at time t, $M(t) - \Delta M$, whose fuel is not ejected as exhaust in the next time interval Δt, and II) the amount of fuel ejected as exhaust in the next

[1]That is, as exact as we can given the many approximations we have made in this first analysis of the problem.

time interval Δt, being ΔM: We'll evaluate the equation from the earth-based reference frame.

$$\sum F_y \Delta t = \Delta p_{\text{system}} = \Delta p_{\text{rocket}} + \Delta p_{\text{ejected fuel}}\,. \tag{3}$$

where the mass of the 'rocket' is fixed in the duration going from t to $t + \Delta t$, likewise, the mass of the ejected fuel being ejected in the same time interval is fixed. Also, the only external force we are considering for this first approximation is the force of gravity, which we are taking as constant over the flight time under consideration. So, $\sum F_y = -M_{\text{system}} g$.

Thus, (3) becomes

$$\begin{aligned} -M(t) g \Delta t &= [M(t) - \Delta m]\Delta V + \Delta m \Delta v \\ &= M(t)\Delta V + \Delta m \Delta v\,, \end{aligned} \tag{4}$$

where Δm and ΔV are both small quantities, but Δv is **not** a small quantity, and we have retained terms only first-order small. Now, since the rate at which fuel is ejected from the rocket is a constant (1),

$$\Delta m = -\kappa \Delta t\,, \tag{5}$$

and, since, (with v_f equal the final velocity of the ejected fuel [in the form of exhaust] and v_i equal to the initial velocity of the ejected fuel)

$$\Delta v = v_f - v_i = (V - \nu) - V = -\nu\,, \tag{6}$$

where v_f and v_i are understood as velocities with respect to the earth-fixed reference frame, over the short time interval Δt.[2] Then (4) becomes

$$-Mg\Delta t = M\Delta V - \kappa\nu\Delta t\,, \tag{7}$$

where we have ignored the second-order small term $\Delta m \Delta V$. So, on letting $\Delta t \mapsto dt$, we get

$$-Mgdt = MdV - \kappa\nu dt\,. \tag{8}$$

We can reduce the number of times M appears in this equation by dividing through by M:

$$-gdt = dV - \frac{\kappa\nu dt}{M}\,. \tag{9}$$

On remembering that $dM = -\kappa dt$ and solving the above for dV, we can write

$$dV = \nu\frac{dM}{M} - gdt\,. \tag{10}$$

Therefore, the differential equation we were asked to find is

$$\dot{V} = \nu\frac{\kappa}{M} - g\,. \tag{11}$$

[2] We already knew this to be the case because the change in speed of the mass represented by Δm will be the same as determined by any inertial reference frame.

The easiest way to integrate this equation is to go back to (10) and integrate from $t = 0$ to $t = t$, to get

$$\int_0^t dV = \nu \int_{M_0}^{M_t} \frac{dM}{M} - \int_0^t g dt\,, \tag{12}$$

or,

$$V \Big|_0^t = -\nu \ln M \Big|_{M_0}^{M_t} - gt\Big|_0^t\,. \tag{13}$$

Yielding,

$$V(t) = V_0 + \nu \ln \frac{M_0}{M(t)} - gt\,. \tag{14}$$

Our final simplification is to set $V_0 = 0$ at rocket lift-off, and get:

$$V(t) = \nu \ln \frac{M_0}{M(t)} - gt\,, \tag{15}$$

where

$$M(t) = M_0 - \kappa t\,, \tag{16}$$

and where, as you remember, M_0 is the initial mass of both the rocket and its fuel.

Last question: How long will the fuel last? Let T be the time when the fuel runs out. Let M_R be the mass of the rocket with no fuel left in it. Then $M_R = M(T) = M_0 - \kappa T$. Solving for T, we get

$$T = \frac{M_0 - M_R}{\kappa}\,. \tag{17}$$

4 Conclusion

So, there you have it: My attempt to produce the rocket equation for strictly vertical flight, while ignoring atmospheric affects, non-inertial effects, and the changing of the value of g as the rocket attains higher elevations.

I wonder if the Flat-Earthers even believe in the $-gt$ term on the RHS of (15), since most of them that I've seen on YouTube claim to **not** believe in gravity. Therefore, they should be able to get their rockets really high up there without gravity to slow them down.