Adv. Quantum Mechanics Notes for L. Susskind's Lecture Series (2013), Lecture 1

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Abstract

This paper contains my notes on Lecture One of Leonard Susskind's 2013 presentation on Advanced Quantum Mechanics for his Stanford Lecture Series. These notes are meant to aid the viewer in following Susskind's presentation, without having to take copious notes. This time we do a short review of introductory quantum mechanics and then move on to more advanced topics.[1]

1 A Little Review

We represent quantum states in the 'ket' form as $|\,\psi\,\rangle$ or in the 'bra' form as $\langle\,\psi\,|$. These vectors are complex conjugates of each other: $|\,\psi\,\rangle$ is a column vector and $\langle\,\psi\,|$ is a row vector. Observables are the things we can measure. They are represented by linear hermitian operators. These operators A act on the states and they satisfy the relation

$$A^\dagger = A\,, \tag{1}$$

where the dagger represents hermitian conjugation.

Let $|\,\alpha\,\rangle$ be a state vector and let A act on this such that

$$A\,|\,\alpha\,\rangle = \alpha\,|\,\alpha\,\rangle\ , \tag{2}$$

then α is said to be an *eigenvalue* of operator A and $|\,\alpha\,\rangle$ is an *eigenvector* of A.[2]

As an axiom of quantum mechanics, these eigenvalues are the possible values that could be measured in an experiment on a quantum system. The state that the system is in, immediately after measurement α has been made, is $|\,\alpha\,\rangle$.

For every ket vector $|\,\phi\,\rangle$, we define its corresponding bra vector by

$$\langle\,\phi\,| \equiv |\,\phi\,\rangle^\dagger\ . \tag{3}$$

For every pair of ket vectors, $|\,\psi\,\rangle$ and $|\,\phi\,\rangle$, we define their *inner product* by

$$\langle\,\phi\mid\psi\,\rangle\ , \tag{4}$$

which is a complex number. Generally, this inner product is not a simple dot product because these two vectors in the ket forms are not usually so simple as being $n\times 1$ matrices. But, like as with

[1] I expect to reference L. Susskind's book *Quantum Mechanics — The Theoretical Minimum* (Susskind and Friedman, Basic Books, 2014) from time to time, and I will refer to it by QMTTM.

[2] This nomenclature comes out of fundamental linear algebra, not quantum mechanics, per se.

the dot product between $|\,\psi\,\rangle$ and $|\,\phi\,\rangle$, their inner product is zero if and only if they are mutually *orthogonal*. And if their inner product is zero, their two states are experimentally distinguishable from each other.

The coordinate measurable

So, now we deal with the points on the x axis, which represent a continuous variable, say x. Physically speaking, these points along the x axis are places that a quantum particle could exist at some moment. If two points on the x axis are distinguishable, such as is the case of points x and x' as depicted in Fig. 1, then

$$\langle\, x' \mid x \,\rangle = 0\,. \tag{5}$$

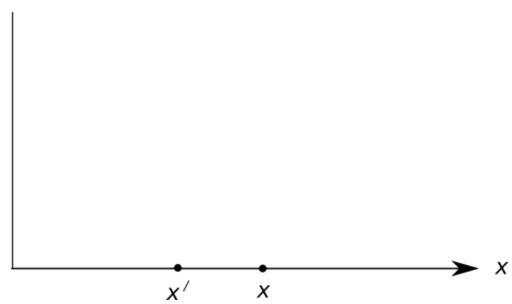

Figure 1. Points x and x' are distinguishable, hence their states are distinguishable. This has mathematical consequences in QM.

Now, let $|\,\psi\,\rangle$ be any state of some particle, then its inner product with $|\,x\,\rangle$ is given as

$$\langle\, x \mid \psi \,\rangle = \psi(x) \tag{6}$$

and is called the *wave function* of the Schrödinger theory. We associate with $\psi(x)$ the probability of finding the particle at location x by

$$P(x) = \langle\, \psi \mid \psi \,\rangle = \psi(x)^*\psi(x)\,, \tag{7}$$

which is a non-negative real number.

As an operator, x acts on wave functions in the obvious manner:

$$x\psi(x)\,, \tag{8}$$

the eigenfunctions of this operator being Dirac delta functions. If a particle is known to be at point x_0 then

$$x\delta(x - x_0) = x_0\delta(x - x_0)\,. \tag{9}$$

Finally, this one-dimensional case generalizes to the three-dimensional case in the obvious way.

The momentum observable

The momentum observable needs a momentum operator, which we'll call P, and

$$P\psi(x) = -i\hbar\frac{\partial\psi(x)}{\partial x}\,. \tag{10}$$

So, we wish to know the eigenvalue/eigenvector states for the momentum operator. Let's set it up formally, as

$$P \,|\, \psi(x) \,\rangle = p_0 \,|\, \psi(x) \,\rangle \ , \tag{11}$$

which translates to

$$-i\hbar \frac{\partial \psi(x)}{\partial x} = p_0 \psi(x)\,. \tag{12}$$

For x one-dimensional, we get the easy solution for $\psi(x)$ as

$$\psi(x) = \exp\{ip_0 x/\hbar\}\,. \tag{13}$$

Transformations on states

We study here two types of transformations of states: one kind is time-independent and the other is time-dependent, which we'll represent by the symbol $U(t)$, which stands for *unitary.* $U(t)$ performs a continuous translation in time (called a *time-evolution operator*), so that

$$U(t) \,|\, \psi(0) \,\rangle = |\, \psi(t) \,\rangle \ , \tag{14}$$

or, more generally,

$$U(t) \,|\, \psi(t_1) \,\rangle = |\, \psi(t_1 + t) \,\rangle \ . \tag{15}$$

We'll now show that $U(t)$ is not a hermitian operator. Then what is it? We need or rule to help us here:

Quantum states evolve in time so that distinguishable states remain distinguishable.

Remember that absolute distinguishability of states of two particles requires that the states are (mutually) orthogonal.[3]

For arbitrary vectors (in the Hilbert space) $|\, \psi \,\rangle$ and $|\, \phi \,\rangle$ at time now, their inner product is $\langle\, \phi \mid \psi \,\rangle$. Now, we let them both evolve in time by

$$|\, \psi \,\rangle \longrightarrow U(t) \,|\, \psi \,\rangle \qquad |\, \phi \,\rangle \longrightarrow U(t) \,|\, \phi \,\rangle \ . \tag{16}$$

To prepare for the inner product of these two time-evolved vectors, we need the hermitian conjugate of $U \,|\, \phi \,\rangle$, that is (suppressing the explicit t),

$$(U \,|\, \phi \,\rangle)^\dagger = \langle\, \phi \,|\, U^\dagger\,. \tag{17}$$

Therefore,

$$\langle\, \phi \mid \psi \,\rangle \longrightarrow \langle\, \phi \,|\, U^\dagger U \,|\, \psi \,\rangle = \langle\, \phi \mid \psi \,\rangle \ . \tag{18}$$

This implies that

$$U^\dagger U = I\,, \tag{19}$$

where I is the identity operator. One way to interpret (18) is that inner products do not change in time. We say that operator X is a symmetry operator if satisfies[4]

$$X^\dagger X = I\,, \tag{20}$$

[3] Assuming that the above rule is consistent with what Prof. Susskind wanted to convey, he referred to it on page 97 of QMTTM as the 'conservation of distinctions'.

[4] From here on, a symmetry operator shall mean other than a unitary operator.

where X may or may not be time dependent. In the case of U in (19), it is time dependent.

For contrast, neither x nor p is a unitary operator. One characteristic of a unitary operator is that its eigenvalue is a phasor, that is, a complex number c of unit absolute value. That is, $|\,c\,| = 1$ where $|\,c\,|^2 = c^*c$. So, let X be a unitary operator which acts on an eigenvector $|\,\phi\,\rangle$, to give

$$X\,|\,\phi\,\rangle = c\,|\,\phi\,\rangle \ . \tag{21}$$

Taking the hermitian conjugate of both sides of this, we get

$$(X\,|\,\phi\,\rangle)^\dagger = \langle\,\phi\,|\,X^\dagger = (c\,|\,\phi\,\rangle)^\dagger = \langle\,\phi\,|\,c^* \,. \tag{22}$$

On taking the inner product of this vector with itself (and using that $X^\dagger X = I$), we get

$$\langle\,\phi\,|\,X^\dagger X\,|\,\phi\,\rangle = \langle\,\phi\,|\,\phi\,\rangle = \langle\,\phi\,|\,c^*c\,|\,\phi\,\rangle \ . \tag{23}$$

Therefore, $c^*c = 1$. So, c is a point on the unit circle of the complex plane.

Reasonable Assumption 1: If U is a unitary operator on a vector space, that is responsible for a smooth evolution of states at one time to corresponding states at another time, then in an instant of time, no change can occur, leading us to the result

$$U(0) = I\,, \tag{24}$$

where, again, I is the identity operator. Our next step is to consider $U(\epsilon)$ where ϵ is a small quantity, then,

$$U(\epsilon) \approx I + \epsilon G\,, \tag{25}$$

where G is itself an operator. We can find out something about G by multiplying (25) through by its hermitian conjugate,

$$\begin{aligned} I = U(\epsilon)^\dagger U(\epsilon) &= (I+\epsilon G)^\dagger (I+\epsilon G) = (I+\epsilon G^\dagger)(I+\epsilon G) \\ &= I + \epsilon(G^\dagger + G)\,, \end{aligned} \tag{26}$$

where we have kept terms of first order and less in powers of ϵ. Hence, we conclude that

$$G^\dagger = -G\,. \tag{27}$$

The operator G is analogous to a pure imaginary number.[5] We are free to replace G by $-iH$, where H is a hermitian operator, and, H is some observable.

Now,

$$U(t) \approx I - itH \longrightarrow e^{-itH}\,. \tag{28}$$

For small ϵ,

$$|\,\psi(t+\epsilon)\,\rangle = e^{-i\epsilon H}\,|\,\psi(t)\,\rangle = (1 - i\epsilon H)\,|\,\psi(t)\,\rangle \ , \tag{29}$$

hence

$$|\,\psi(t+\epsilon)\,\rangle - |\,\psi(t)\,\rangle = -i\epsilon H\,|\,\psi(t)\,\rangle \ , \tag{30}$$

or

$$\frac{|\,\psi(t+\epsilon)\,\rangle - |\,\psi(t)\,\rangle}{\epsilon} = -iH\,|\,\psi(t)\,\rangle \ . \tag{31}$$

From this we can conclude that, when converted from state vector form to wave function form,

$$\frac{\partial\psi(t)}{\partial t} = -iH\psi(t)\,, \tag{32}$$

[5] Operators that satisfy (27) are said to be anti-hermitian or skew-hermitian. There is a standard way to convert these into hermitian operators and that is to multiply them by the unit imaginary number i. It's pretty obvious that if G is antihermitian then iG is hermitian and $-iG$ is also hermitian.

which is the time-dependent Schrödinger Equation and H is recognized as the Hamiltonian of the system. But if we prefer, we can write (32) as

$$\frac{\partial \,|\, \psi(t) \,\rangle}{\partial t} = -iH \,|\, \psi(t) \,\rangle \ . \tag{33}$$

Susskind remarked that he needed to put $\hbar$ back into the equation, which would give us

$$\hbar \frac{\partial \,|\, \psi(t) \,\rangle}{\partial t} = -iH \,|\, \psi(t) \,\rangle \ . \tag{34}$$

(This is one of those aspects of concern when one sets $\hbar$ equal to unity. It might be more efficient to set unity equal to $\hbar$.)

Okay, what about the time-independent Schrödinger Equation? In this case, let $|\, \psi \,\rangle$ be an eigenvector of H then

$$H \,|\, \psi \,\rangle = E \,|\, \psi \,\rangle \ , \tag{35}$$

where E is the corresponding eigenvalue of H. We can recast this into a more mnemonic form as

$$H \,|\, E \,\rangle = E \,|\, E \,\rangle \ . \tag{36}$$

Thus we've reached the end of the review of intro quantum mechanics.

2 More advanced topics

Now that we've finished our review of unitary operators, which are the time-dependent operator that leave inner products invariant, what other operators (i.e., time-independent operators) that leave inner products invariant?

Let U represent a unitary operator. Let V represent some other operator, such as a rotation operator (that is, a symmetry operation). Then, we take as definitions,

$$\begin{aligned} U \,|\, \psi_1 \,\rangle &= |\, \psi_2 \,\rangle \\ V \,|\, \psi \,\rangle &= |\, \psi' \,\rangle \ . \end{aligned} \tag{37}$$

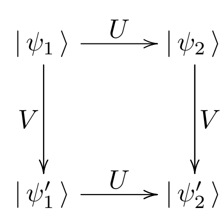

Figure 2. A commutative diagram for the operators U and V as they act on state vectors.

The diagram above is called a "commutative diagram" by mathematicians. What it means is that the 'direction' of movement through the diagram from top left to bottom right does not matter when going from $|\, \psi_1 \,\rangle$ to $|\, \psi_2' \,\rangle$, or, in other words

$$UV \,|\, \psi_1 \,\rangle = |\, \psi_2' \,\rangle = VU \,|\, \psi_1 \,\rangle \ , \tag{38}$$

with the conclusion that

$$UV = VU \tag{39}$$

is an equality of operators. The upshot of this is that symmetry operators of a system commute with unitary operators of the system.

Anyway, we're going to construct our symmetry operators analogously to how we constructed the unitary operators. Let V stand for a symmetry operator, which, of course, means that it must commute with the system Hamiltonian H:

$$[\,H, V\,] = 0\,, \tag{40}$$

that can be parametrized by a continuous variable, which, for the moment, we'll call ϵ. Taking ϵ as small, $V(\epsilon)$ can be expanded approximately as

$$V(\epsilon) \approx 1 - i\epsilon G\,, \tag{41}$$

where G is called the *generator* of the symmetry. Following a similar argument to proving that the H associated with unitary operator is hermitian, we conclude that the iG in (41) is also hermitian.

Now, starting with (40), we get,

$$[\,H, 1 - i\epsilon G\,] = [\,H, -i\epsilon G\,] = -i\epsilon[\,H, G\,] = 0\,. \tag{42}$$

From this we conclude that

$$[\,H, G\,] = 0\,. \tag{43}$$

The upshot of this is that the generators of symmetry operators commute with the Hamiltonian. This suggests that one way to find the symmetries of a Hamiltonian is to find the set of all generators that commute with the Hamiltonian, and this is a purely mathematical problem.

However, another way to look at the problem of finding the symmetries of a system is to look for all the conserved properties of continuous symmetries of a system.

Now for some examples:

If we begin with a wave function $\psi(x)$ what operation must we perform on it to translate it to $\psi(x-\epsilon)$? Let V be the operator that accomplishes this:

$$V\psi(x) = \psi(x-\epsilon) \approx \psi(x) - \frac{\partial\psi(x)}{\partial x}\epsilon\,. \tag{44}$$

Hence,

$$V\psi(x) = \big(1 - \epsilon\frac{\partial}{\partial x}\big)\psi(x)\,. \tag{45}$$

Or, more abstractly,

$$V = 1 - \epsilon\frac{\partial}{\partial x}\,. \tag{46}$$

But, as we remember,

$$P = -i\hbar\frac{\partial}{\partial x}\,, \tag{47}$$

where P is the momentum operator. Therefore,

$$V = 1 - \frac{i\epsilon}{\hbar}P\,. \tag{48}$$

Thus, P is now seen as the generator for the infinitesimal operator V. But, since we restricted our analysis to just the x dimension, we should also note this by referencing the momentum generator by P_x.

For a free particle moving in the x direction, the Hamiltonian is given by

$$H = \frac{P_x^2}{2m}\,. \tag{49}$$

Therefore,

$$[\,H, P_x\,] = \Big[\,\frac{P_x^2}{2m}, P_x\,\Big] = 0\,. \tag{50}$$

Hence, we conclude that the momentum in the x direction is conserved, and we've revealed a translation symmetry. If we generalize (49) to

$$H = \frac{P_x^2}{2m} + \frac{P_y^2}{2m} + \frac{P_z^2}{2m}\,, \tag{51}$$

we get that momentum is conserved on all three directions at once.

Next time, we'll cover rotational symmetry.