Adv. Quantum Mechanics Notes for L. Susskind's Lecture Series (2013), Lecture 2

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Abstract

This paper contains my notes on Lecture Two of Leonard Susskind's 2013 presentation on Advanced Quantum Mechanics for his Stanford Lecture Series. These notes are meant to aid the viewer in following Susskind's presentation, without having to take copious notes. This time we begin by studying energy degeneracies in discrete systems.

1 Rotation symmetry

We begin by considering a particle constrained to move in a circle. Let this circle be in the x, y plane. Now, since the particle cannot change its distance from the origin of coordinates, we can alternatively follow the motion of the particle by means of its angle θ from the x axis.

This particle has a wave function given by $\psi(\theta)$. Now, a counterclockwise rotation does the following

$$\psi(\theta) \longrightarrow \psi(\theta - \epsilon)\,, \tag{1}$$

where ϵ is a small parameter. To first order in ϵ,

$$\delta\psi = -\epsilon \frac{\partial \psi}{\partial \theta} = -i\epsilon\Big(-i \frac{\partial \psi}{\partial \theta}\Big)\,. \tag{2}$$

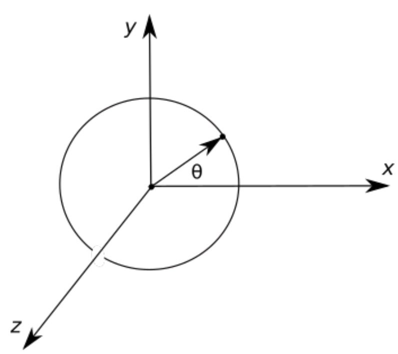

Figure 1. The rotation of a particle in the x, y plane as viewed in 3-space.

We'll give a name to the angular derivative depicted in (2)

$$L \equiv -i\hbar \frac{\partial}{\partial \theta}\,, \tag{3}$$

where we have included the $\hbar$ factor. L is knowwn as the angular momentum operator. Then (2) becomes

$$\delta\psi = -\frac{i\epsilon}{\hbar} L\psi\,. \tag{4}$$

Proceeding as we did for linear momentum,

$$L\,|\,\psi\,\rangle = m\,|\,\psi\,\rangle\ , \tag{5}$$

where, by convention, m is the z component of the angular momentum.

$$-i\hbar\frac{\partial}{\partial\theta}\psi = m\psi\,. \tag{6}$$

The solution for ψ is given as

$$\psi(\theta) = e^{im\theta/\hbar}\,. \tag{7}$$

But we have an additional constraint that

$$\psi(2\pi) = \psi(0)\,. \tag{8}$$

Now, at $\theta = 0$, $\psi(0) = 1$. And at $\theta = 2\pi$

$$e^{2im\pi/\hbar} = 1\,. \tag{9}$$

Hence, $m/\hbar$ must be an integer. Then, the quantization of angular momentum in units of $\hbar$ is

$$L = m\hbar\,. \tag{10}$$

With respect to how the energy of the revolving particle goes as it is dependent on m, we expect that

$$E(-m) = E(m)\,. \tag{11}$$

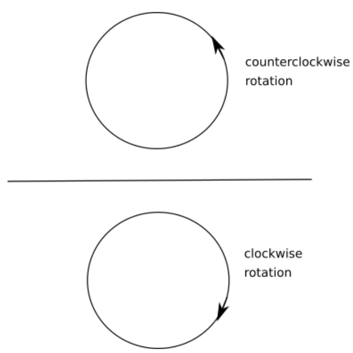

Figure 2. The equality of $E(-m)$ and $E(m)$ is founded on the reflection symmetry about the horizontal line, evident in this figure.

Let's formalize this reflection symmetry by introducing a reflection operator M such that

$$M\psi(\theta) = \psi(-\theta)\,, \tag{12}$$

where we can reference the θ in Fig. 1.

Let's consider the commutator of M and L.

$$[\,M, L\,]e^{im\theta} = 2me^{im\theta}\,. \tag{13}$$

Proof:

$$MLe^{im\theta} = Mme^{im\theta} = me^{-im\theta}\,, \tag{14}$$

$$LMe^{im\theta} = Lme^{-im\theta} = -me^{-im\theta}\,. \tag{15}$$

So, although M and L both commute with the Hamiltonian, they do not commute with each other.

Now we look at the space of two generators A, B of symmetries. This means they they both commute with the Hamiltonian.

$$[\,A, H\,] = 0 \quad \text{and} \quad [\,B, H\,] = 0\,. \tag{16}$$

If we let

$$[\,A, B\,] = iC\,, \tag{17}$$

where C is a hermitian operator, then does C commute with H?

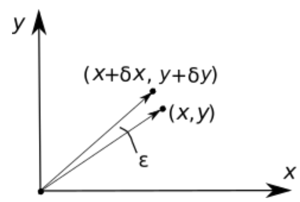

Figure 3. A particle moves from point (x, y) to point $(x + \delta x, y + \delta y)$

Let (x, y) and $(x + \delta x, y + \delta y)$ be nearby points in the x, y plane, being of angular separation ϵ, which is a small angle, as seen in Fig. 3. Then

$$\begin{aligned} \delta x &= -\epsilon y\,, \\ \delta y &= \epsilon x\,. \end{aligned}$$

Then

$$\begin{aligned} \delta\psi &= \frac{\partial\psi}{\partial x}\delta x + \frac{\partial\psi}{\partial y}\delta y \\ &= -\epsilon\frac{\partial\psi}{\partial x}y + \epsilon\frac{\partial\psi}{\partial y}x \\ &= i\epsilon L_z\psi\,. \end{aligned} \tag{18}$$

Now, given that the partial derivatives are proportional to momenta, we get

$$\delta\psi = -i\epsilon y p_x + i\epsilon x p_y\,. \tag{19}$$

Finally, we get

$$L_x = yp_z - zp_y\,, \tag{20a}$$

$$L_y = zp_x - xp_z\,, \tag{20b}$$

$$L_z = xp_y - yp_x\,. \tag{20c}$$

These L operators are symmetries if the system is rotationally invariant, such as a particle moving in a central force field. So, if we replace the Latin subscripts by an index i in these last equations, we have that

$$[\, L_i, H \,] = 0, \quad i = 1, 2, 3\,. \tag{21}$$

Now, we would like to know the values of $[\, L_i, L_j \,]$ for $i \neq j$. So, we had better investigate the commutation relations among the coordinates and momenta, first:

$$[\, x, y \,] = 0, \ldots, \quad [\, p_x, p_y \,] = 0, \ldots \quad . \tag{22}$$

Which objects don't commute with each other?

$$[\, x, p_x \,] = i\hbar\,, \tag{23a}$$
$$[\, y, p_y \,] = i\hbar\,, \tag{23b}$$
$$[\, z, p_z \,] = i\hbar\,. \tag{23c}$$

But

$$[\, x_i, p_j \,] = 0 \quad \text{when} \quad i \neq j\,. \tag{24}$$

Now,

$$\begin{aligned}
[\, L_x, L_y \,] &= [\, yp_z - zp_y, zp_x - xp_z \,] \\
&= [\, yp_z, zp_x \,] + [\, zp_y, xp_z \,] \\
&= -iyp_x + ixp_y \\
&= iL_z\,.
\end{aligned} \tag{25}$$

By similar reasoning, we get

$$[\, L_y, L_z \,] = iL_x\,, \tag{26}$$
$$[\, L_z, L_x \,] = iL_y\,. \tag{27}$$

Let's now look carefully at the eigenvalue equations of the angular momentum operators acting on eigenstates.

$$L_z \,|\, m \,\rangle = m \,|\, m \,\rangle\ . \tag{28}$$

We want to know the possible values of m. We are going to go about answering this question quite cleverly. First, we introduce L_+ and L_-.

$$\begin{aligned}
L_+ &= L_x + iL_y \\
L_- &= L_x - iL_y
\end{aligned} \tag{29}$$

Then

$$[\, L_+, L_z \,] = -L_+\,, \tag{30}$$
$$[\, L_-, L_z \,] = \quad L_-\,. \tag{31}$$

Now, suppose the m in (28) is a known value. We can use this to find another eigenvalue. From (30) we get

$$[\, L_+, L_z \,] \,|\, m \,\rangle = (L_+L_z - L_zL_+) \,|\, m \,\rangle = -L_+ \,|\, m \,\rangle\ . \tag{32}$$

Then

$$L_+L_z \,|\, m \,\rangle - L_zL_+ \,|\, m \,\rangle = -L_+ \,|\, m \,\rangle\ . \tag{33}$$

And then

$$(m+1)L_+ \,|\, m \,\rangle = L_z L_+ \,|\, m \,\rangle \ . \tag{34}$$

Therefore, $L_+ \,|\, m \,\rangle$ is an eigenvector of operator L_z with eigenvalue $m+1$. Similarly,

$$(m-1)L_- \,|\, m \,\rangle = L_z L_- \,|\, m \,\rangle \ . \tag{35}$$

We now have a procedure to elicit a spectrum of eigenvalue, given a single eigenvalue.

Next, we operate with the Hamiltonian

$$H \,|\, m \,\rangle = E \,|\, m \,\rangle \ . \tag{36}$$

But, we now know that $L_+ \,|\, m \,\rangle$ is also an eigenvector of L_z, so

$$HL_+ \,|\, m \,\rangle = L_+ H \,|\, m \,\rangle = EL_+ \,|\, m \,\rangle \ . \tag{37}$$

But we can rewrite this as

$$H \,|\, m+1 \,\rangle = E \,|\, m+1 \,\rangle \ . \tag{38}$$

But this energy E is the same energy of Eq. (36), which means that we've arrived at a degeneracy: multiple states have the same energy.

"This is the idea of degeneracies following from symmetry, when symmetries don't commute with each other."

Our procedure is the quantum analogue of rotating the L_z axis in 3-space and thereby manifesting a degeneracy.

We'll be meeting raising and lowering operators in other places in quantum theory.