Adv. Quantum Mechanics Notes for L. Susskind's Lecture Series (2013), Lecture 3

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Abstract

This paper contains my notes on Lecture Three of Leonard Susskind's 2013 presentation on Advanced Quantum Mechanics for his Stanford Lecture Series. These notes are meant to aid the viewer in following Susskind's presentation, without having to take copious notes.

1 Angular Momentum

This time we begin by looking at the eigenstates of the angular momentum operator in two dimensions, in terms of the radius r and angle θ.

$$L \equiv -i\hbar \frac{\partial}{\partial \theta} \,. \tag{1}$$

Next, we set up the eigenvalue equation

$$-i\hbar \frac{\partial}{\partial \theta} \psi(r, \theta) = \ell \psi(r, \theta) \,, \tag{2}$$

where r will play no role here. Anyway, the solution is

$$\psi = e^{i\ell\theta/\hbar} \chi(r) \,, \tag{3}$$

where $\chi(r)$ is unspecified at this point, and where ℓ has to be an integer.

On moving up to three spatial dimensions, we have

$$\psi(r, \theta, \phi) = \chi(r) Y(\theta, \phi) \,. \tag{4}$$

However, we don't want to worry about the complications that attend both θ and ϕ at the same time, so we'll use some tricks we invoked the last lecture.

As you recall, we started off with the angular momentum operators, L_x, L_y, L_z and found that they obey the commutation relations,

$$\begin{aligned} [\, L_x, L_y \,] &= i\hbar L_z \\ [\, L_y, L_z \,] &= i\hbar L_x \,, \\ [\, L_z, L_x \,] &= i\hbar L_y \,. \end{aligned}$$

Then we chose to adopt L_z as special among these three as an operator on states.

$$L_z \,|\, m \,\rangle = m \,|\, m \,\rangle \,, \tag{5}$$

where m is the eigenvalue of L_z and $|\, m \,\rangle$ is its corresponding eigenvector. And we defined

$$L_\pm = L_x \pm i L_y \,. \tag{6}$$

We found that

$$L_+ \,|\, m \,\rangle = |\, m+1 \,\rangle \,. \tag{7}$$

This tells us that L_+ is a raising operator. Given some eigenstate $|\, m \,\rangle$, L_+ produces the eigenstate $|\, m+1 \,\rangle$, but we don't know from the analysis so far if m is integer valued or not. Similarly,

$$L_- \,|\, m \,\rangle = |\, m-1 \,\rangle \,. \tag{8}$$

Question: Is there a maximum step we can go up this ladder? There can be. This will happen if there exists an m such that $L_+ \,|\, m \,\rangle = 0$.

Now, we make a symmetry argument: Because we can rotate the z axis so that the $+z$ and $-z$ directions are swapped, we can infer that the behavior of the eigenvalues must be symmetric about the $z = 0$ point on the z axis. In which case, if there is a maximum eigenvalue, there must also be a minimum eigenvalue. We'll now prove that m must be an integer in the appropriate case. Let the maximum eigenvalue be denoted as $\overline{m}$. If we start at $-\overline{m}$, we can arrive at $\overline{m}$ by adding to it some integer number of steps $2w$ or just w, each step being ± 1. For the former case, we get

$$\overline{m} = -\overline{m} + 2w \,. \tag{9}$$

Solving this for $\overline{m}$, we get $\overline{m} = w$; hence, $\overline{m}$ is an integer, which forces all the m's to be integer valued. In the other case, we get

$$\overline{m} = -\overline{m} + w \,, \tag{10}$$

implying that $\overline{m} = w/2$. For the case $w = 1$, we get $\overline{m} = 1/2$ and $-\overline{m} = -1/2$ For the time being, we'll ignore the half-integer case.

Now, in standard terminology, the maximum value of m, $\overline{m}$, is referred to as ℓ. Likewise, the minimum value of m is $-\ell$, for a total of $2\ell + 1$ states, called a *multiplet*.

Next, we examine the operator L^2:

$$\begin{aligned} L^2 &= L_x^2 + L_y^2 + L_z^2 \\ &= L_z^2 + (L_x - iL_y)(L_x + iL_y) - i[\, L_x, L_y \,] \,, \end{aligned} \tag{11}$$

where the last term accounts for the lack of commutivity of L_x and L_y. So, on simplifying, we get

$$L^2 = L_z^2 + L_z + L_- L_+ \,. \tag{12}$$

Let's denote the highest state by $|\, \ell \,\rangle$, then $L_+ \,|\, \ell \,\rangle = 0$. Now, consider the following:

$$\begin{aligned} L^2 \,|\, \ell \,\rangle &= L_z^2 \,|\, \ell \,\rangle + L_z \,|\, \ell \,\rangle + \cancelto{0}{L_- L_+ \,|\, \ell \,\rangle} \\ &= \ell^2 \,|\, \ell \,\rangle + \ell \,|\, \ell \,\rangle \\ &= \ell(\ell+1) \,|\, \ell \,\rangle \,. \end{aligned} \tag{13}$$

The following are a few results we'll need from time to time:

$$[\, L^2, L_x \,] = 0, \quad [\, L^2, L_y \,] = 0, \quad [\, L^2, L_z \,] = 0 \,. \tag{14}$$

Let's show this for the case $[\,L^2, L_x\,] = 0$ (with some virtual emplacement).

$$\begin{aligned}
[\,L^2, L_x\,] &= [\,L_x^2 + L_y^2 + L_z^2, L_x\,] = [\,L_y^2 + L_z^2, L_x\,] \\
&= [\,L_y^2, L_x\,] + [L_z^2, L_x\,] \\
&= L_y L_y L_x - L_x L_y L_y + L_z L_z L_x - L_x L_z L_z \\
&= L_y L_y L_x - L_y L_x L_y + L_y L_x L_y - L_x L_y L_y + L_z L_z L_x - L_x L_z L_z \\
&= L_y[\,L_y, L_x\,] + [\,L_y, L_x\,]L_y + L_z L_z L_x - L_z L_x L_z + L_z L_x L_z - L_x L_z L_z \\
&= L_y[\,L_y, L_x\,] + [\,L_y, L_x\,]L_y + L_z[\,L_z, L_x\,] + [\,L_z, L_x\,]L_z \\
&= L_y(-i\hbar L_z) + (-i\hbar L_z)L_y + L_z(i\hbar L_y) + (i\hbar L_y)L_z \\
&= 0\,. \qquad (15)
\end{aligned}$$

In similar fashion, one could prove that $[\,L^2, L_z\,] = 0$.

Now, since L^2 commutes with L_x and L_y singlely, it must also commute with $L_\pm = L_x \pm iL_y$. That is,

$$[\,L^2, L_+\,] = 0, \quad [\,L^2, L_-\,] = 0\,. \qquad (16)$$

We wish to show that the result in (13) is not unique to the highest angular momentum state ℓ. For example,

$$\begin{aligned}
L^2\,|\,\ell - 1\,\rangle &= L^2 L_-\,|\,\ell\,\rangle \\
&= L_- L^2\,|\,\ell\,\rangle \\
&= \ell(\ell+1)L_-\,|\,\ell\,\rangle \\
&= \ell(\ell+1)\,|\,\ell - 1\,\rangle\;. \qquad (17)
\end{aligned}$$

Let's repeat this proof in a more general case. We wish to show that $L^2\,|\,\ell - n\,\rangle = \ell(\ell+1)\,|\,\ell - n\,\rangle$ for $0 \le n \le \ell$, then

$$\begin{aligned}
L^2\,|\,\ell - n\,\rangle &= L^2(L_-)^n\,|\,\ell\,\rangle \\
&= (L_-)^n L^2\,|\,\ell\,\rangle \\
&= \ell(\ell+1)(L_-)^n\,|\,\ell\,\rangle \\
&= \ell(\ell+1)\,|\,\ell - n\,\rangle\;. \qquad (18)
\end{aligned}$$

Hence,

$$\begin{aligned}
L_z\,|\,m\,\rangle &= m\,|\,m\,\rangle \\
L^2\,|\,\ell\,\rangle &= \ell(\ell+1)\,|\,\ell\,\rangle\;. \qquad (19)
\end{aligned}$$

For a central force, the angular momentum commutes with the Hamiltonian, which is enough to prove that. Therefore, if you apply L to a function, it doesn't change the energies.

2 Central Forces

We won't be entirely rigorous in our treatment here, instead, relying on intuition and some insight from classical physics. What we want to do is to solve the Schrödinger equation for a radially symmetric potential.

$$H = \frac{\mathbf{P}^2}{2m} + V(r)\,. \tag{20}$$

Classically, we know that angular momentum is conserved because we have rotational invariance in the potential.

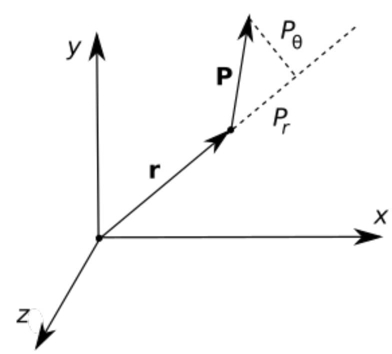

Figure 1. The rotation of a particle in the x, y plane as viewed in 3-space.

Returning to (20), we have that

$$H = \frac{P_r^2 + P_\theta^2}{2m} + V(r)\,. \tag{21}$$

And, classically, the angular momentum goes as

$$\mathbf{L} = \mathbf{r} \times \mathbf{P}\,. \tag{22}$$

Referring to the figure above, we get the scalar form of this as

$$L = r\,P_\theta\,. \tag{23}$$

Or,

$$L^2 = r^2 P_\theta^2\,. \tag{24}$$

With this, the Hamiltonian becomes

$$H = \frac{P_r^2}{2m} + \frac{1}{2m}\frac{L^2}{r^2} + V(r)\,. \tag{25}$$

The last two terms of this last equation are often referred to as the *effective potential.*

Now, moving to the quantum version,

$$H\psi = E\psi\,, \tag{26}$$

$$\frac{(i\hbar)^2}{2m}\frac{\partial^2}{\partial r^2}\psi + \frac{1}{2m}\frac{L^2}{r^2}\psi + V(r)\psi = E\psi\,, \tag{27}$$

or

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2}\psi(r)+\frac{1}{2m}\frac{\ell(\ell+1)}{r^2}\psi(r)+V(r)\psi(r)=E\psi(r)\,. \tag{28}$$

Now, a *node* of the wave function is a point on the x axis that the function cross it. The number of nodes characterize the different energy levels of the Schrödinger equation. The more nodes, the higher the energy. The more nodes, the faster it wiggles. The faster it wiggles, the higher the momentum.

Next, we look more closely at the Schrödinger equation (28). Note that the number ℓ enters into this equation as a parameter.

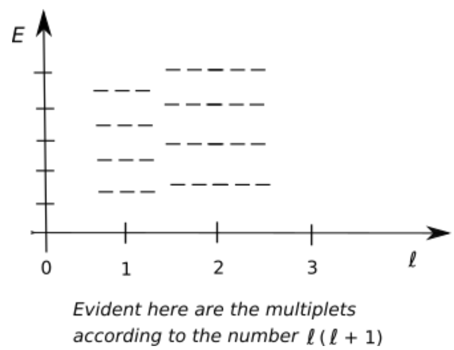

Figure 2. Horizontal lines represent energy levels of this generic quantum system. The dashed line segments represent the number of multiplets per energy level for a given ℓ value.

3 Harmonic Oscillator

We're interested in the case of a mass subject to a linear spring force. This object has a place of equilibrium and when it is displaced a little form that equilibrium point, it will tend to oscillate about the equailibrium point, with or without damping of amplitude.

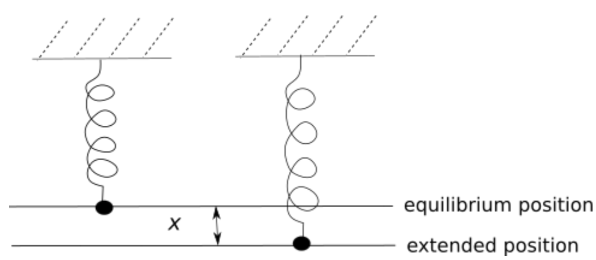

Figure 3. Mass supported by a spring, undergoing harmonic motion. The variable x represents the displacement from the equilibrium position.

In our examination of this setup, we'll take the mass of the suspended object to be unity, and we'll set the spring constant $k=\omega^2$.

Thus,

$$H = \frac{P^2}{2} + \frac{\omega^2 x^2}{2}\,. \tag{29}$$

We'll analyze by Hamilton's equations of motion, given by

$$\frac{\partial H}{\partial P} = \dot{x}\,, \tag{30a}$$

$$\frac{\partial H}{\partial x} = -\dot{P}\,. \tag{30b}$$

Applying (30a) to H, we get

$$P = \dot{x}\,, \tag{31a}$$

$$-\omega^2 x = \dot{P}\,. \tag{31b}$$

Hence,

$$\ddot{x} = \dot{P} = -\omega^2 x\,, \tag{32}$$

which is the equation of an oscillator with frequency ω.

Now we switch to the quantum mechanical viewpoint. Let's go back to (29) and factor it:

$$\begin{aligned} H &= \frac{P^2}{2} + \frac{\omega^2 x^2}{2} \\ &= \tfrac{1}{2}(P + i\omega x)(P - i\omega x) - \frac{i\omega}{2}[\,x, P\,] \\ &= \frac{1}{2}(P + i\omega x)(P - i\omega x) - \frac{i\omega}{2} i \\ &= \frac{1}{2}(P + i\omega x)(P - i\omega x) + \frac{\omega}{2} \\ &= \frac{\omega}{2\omega}(P + i\omega x)(P - i\omega x)\,. \end{aligned} \tag{33}$$

We can ignore the constant term and do some manipulation, to get

$$H = \omega \frac{(P + i\omega x)}{\sqrt{2\omega}} \frac{(P - i\omega x)}{\sqrt{2\omega}}\,. \tag{34}$$

Let's introduce some definitions here:

$$a^+ = \frac{(P + i\omega x)}{\sqrt{2\omega}}\,, \qquad a^- = \frac{(P - i\omega x)}{\sqrt{2\omega}}\,. \tag{35}$$

These two operators are hermitian conjugates of each other. So, with these symbolic simplifications, we can write

$$H = \omega a^+ a^-\,. \tag{36}$$

Let's find the commutator of a^+ and a^-:

$$\begin{aligned} [\,a^+, a^-\,] &= \Big[\,\frac{(P + i\omega x)}{\sqrt{2\omega}}, \frac{(P - i\omega x)}{\sqrt{2\omega}}\,\Big] \\ &= \frac{1}{2\omega}[\,(P + i\omega x), (P - i\omega x)\,] \\ &= \frac{1}{2\omega}\big\{ - i\omega[\,P, x\,] + i\omega[\,x, P\,]\big\} \\ &= -1\,. \end{aligned} \tag{37}$$

Note: I am going to proceed a bit differently than did Susskind at this point, though I'll stay close to it. Actually, I will proceed closer to how this subject was presented in Susskind's book *Quantum Mechanics – The Theoretical Minimum*, Susskind and Friedman, Basic Books (2014), pgs. 327–337.

It is found to be convenient to give a name to a^+a^-.

$$N \equiv a^+a^- \,. \tag{38}$$

The Hamiltonian then becomes

$$H = \omega N \,. \tag{39}$$

Since the product of two hermitian operators is always hermitian, then N is hermitian, thus it has a complete set of eigenvalues and eigenvectors.

So, let $|\, n\,\rangle$ be an eigenstate of N, with equation

$$N\,|\, n\,\rangle = n\,|\, n\,\rangle \,, \tag{40}$$

or

$$a^+a^-\,|\, n\,\rangle = n\,|\, n\,\rangle \,. \tag{41}$$

To aid us in understanding how to proceed,

$$(a^-a^+ - a^+a^-)\,|\, n\,\rangle = |\, n\,\rangle \,. \tag{42}$$

Using (40) and (41), we get

$$(a^-a^+ - n)\,|\, n\,\rangle = |\, n\,\rangle \,, \tag{43}$$

which becomes

$$a^-a^+\,|\, n\,\rangle = (n+1)\,|\, n\,\rangle \,. \tag{44}$$

Multiplying this through by a^+, we get

$$a^+a^-a^+\,|\, n\,\rangle = (n+1)a^+\,|\, n\,\rangle \,. \tag{45}$$

Therefore,

$$Na^+\,|\, n\,\rangle = (n+1)a^+\,|\, n\,\rangle \,. \tag{46}$$

But, on being consistent with (40), we have that

$$N\,|\, n+1\,\rangle = (n+1)\,|\, n+1\,\rangle \,. \tag{47}$$

So, by comparing these last two equation, we must demand that

$$a^+\,|\, n\,\rangle = |\, n+1\,\rangle \,. \tag{48}$$

Hence, a^+ is a raising operator, and, by analogy, a^- is a lowering operator. That is,

$$a^-\,|\, n\,\rangle = |\, n-1\,\rangle \,. \tag{49}$$

Now, since the operator N is positive definite, it cannot have negative eigenvalues, therefore, it must have a least eigenvalue that is at worst zero. The symbol $|\, 0\,\rangle$ is used to represent this state of least eigenvalue. This state is referred to as the *ground state*.

Operationally, the ground state is characterized by

$$a^-\,|\, 0\,\rangle = 0 \,. \tag{50}$$

Final note: We don't have the coefficient of (49) correct. To find it, we start over with

$$a^+ \,|\, n \,\rangle = \alpha_n \,|\, n+1 \,\rangle \ , \tag{51}$$

and solve for α_n. Our constraint is that

$$(\,\langle\, n \,|\, a^-)(a^+ \,|\, n \,\rangle) = (\,\langle\, n+1 \,|\, \alpha_n^*)(\alpha_n \,|\, n+1 \,\rangle)\,, \tag{52}$$

where we used that $(a^+)^* = a^-$. Or

$$(n+1)\,\langle\, n \mid n \,\rangle = |\,\alpha_n\,|^2 \,\langle\, n+1 \mid n+1 \,\rangle \ . \tag{53}$$

On assuming that our eigenvectors are normalized, we get

$$(n+1) = |\,\alpha_n\,|^2 \,. \tag{54}$$

Assuming that α_n is real, we get

$$\alpha_n = (n+1)^{1/2} \,. \tag{55}$$

Therefore, (51) becomes

$$a^+ \,|\, n \,\rangle = (n+1)^{1/2} \,|\, n+1 \,\rangle \ . \tag{56}$$

By similar reasoning, we get

$$a^- \,|\, n \,\rangle = n^{1/2} \,|\, n-1 \,\rangle \ . \tag{57}$$