Adv. Quantum Mechanics Notes for L. Susskind's Lecture Series (2013), Lecture 4

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Abstract

This paper contains my notes on Lecture Four of Leonard Susskind's 2013 presentation on Advanced Quantum Mechanics for his Stanford Lecture Series. These notes are meant to aid the viewer in following Susskind's presentation, without having to take copious notes.

1 Solving for eigenstates of the harmonic oscillator

Returning to the Hamiltonian for an oscillator in 1-D, we have

$$H = \frac{P^2}{2} + \frac{\omega^2}{2}x^2 \,. \tag{1}$$

Now we switch to the quantum mechanical viewpoint. Let's factor this equation:

$$\begin{aligned} H &= \frac{P^2}{2} + \frac{\omega^2 x^2}{2} \\ &= \tfrac{1}{2}(P + i\omega x)(P - i\omega x) - \frac{i\omega}{2}[\,x, P\,] \\ &= \frac{1}{2}(P + i\omega x)(P - i\omega x) - \frac{i\omega}{2} i \\ &= \frac{1}{2}(P + i\omega x)(P - i\omega x) + \frac{\omega}{2} \,. \end{aligned} \tag{2}$$

We can ignore the constant term and do some manipulation, to get

$$H = \omega \frac{(P + i\omega x)}{\sqrt{2\omega}} \frac{(P - i\omega x)}{\sqrt{2\omega}} \,. \tag{3}$$

Note: I will repeat some of the foundational equations that were in the last lecture for use here as well.

Let's introduce some names here:

$$a^+ = \frac{(P + i\omega x)}{\sqrt{2\omega}} \,, \qquad a^- = \frac{(P - i\omega x)}{\sqrt{2\omega}} \,. \tag{4}$$

These two operators are hermitian conjugates of each other. So, with these symbolic simplifications, we can write

$$H = \omega a^+ a^- \,. \tag{5}$$

From the last time, we found that the commutator of a^+ and a^- is:

$$[\, a^-, a^+\,] = 1\,. \tag{6}$$

It is found to be convenient to give a name to a^+a^-.

$$N \equiv a^+a^-\,. \tag{7}$$

N has eigenvalues of $0, 1, 2, \ldots$.

The Hamiltonian then becomes

$$H = \omega(N + \tfrac{1}{2})\,. \tag{8}$$

When $N = 0$, $H = \frac{1}{2}\omega$ is the ground state energy.

So, let $|\, n\,\rangle$ be an eigenstate of N, with equation

$$N\,|\, n\,\rangle = n\,|\, n\,\rangle\ , \tag{9}$$

or

$$a^+a^-\,|\, n\,\rangle = n\,|\, n\,\rangle\ . \tag{10}$$

To aid us in understanding how to proceed,

$$(a^-a^+ - a^+a^-)\,|\, n\,\rangle = |\, n\,\rangle\ . \tag{11}$$

Using (9) and (10), we get

$$(a^-a^+ - n)\,|\, n\,\rangle = |\, n\,\rangle\ , \tag{12}$$

which becomes

$$a^-a^+\,|\, n\,\rangle = (n+1)\,|\, n\,\rangle\ . \tag{13}$$

Multiplying this through by a^+, we get

$$a^+a^-a^+\,|\, n\,\rangle = (n+1)a^+\,|\, n\,\rangle\ . \tag{14}$$

Therefore,

$$Na^+\,|\, n\,\rangle = (n+1)a^+\,|\, n\,\rangle\ . \tag{15}$$

But, on being consistent with (9), we have that

$$N\,|\, n+1\,\rangle = (n+1)\,|\, n+1\,\rangle\ . \tag{16}$$

So, by comparing these last two equation, we must demand that

$$a^+\,|\, n\,\rangle = |\, n+1\,\rangle\ . \tag{17}$$

Hence, a^+ is a raising operator, and, by analogy, a^- is a lowering operator. That is,

$$a^-\,|\, n\,\rangle = |\, n-1\,\rangle\ . \tag{18}$$

By the way, the Schrödinger equation for this problem is

$$-\tfrac{1}{2}\frac{d^2}{dx^2}\psi(x) + \omega^2x^2\psi(x) = E\psi(x)\,. \tag{19}$$

Now, since the operator N is positive definite, it cannot have negative eigenvalues, therefore, it must have a least eigenvalue that is at worst zero. The symbol $|\,0\,\rangle$ is used to represent this state of least eigenvalue. This state is referred to as the *ground state*.

Operationally, the ground state is characterized by

$$a^- \,|\,0\,\rangle = 0\,. \tag{20}$$

It follows immediately that

$$N\,|\,0\,\rangle = 0\,. \tag{21}$$

Question: Can we find the ground state wave function without resorting back to the Schrödinger equation?

Yes, by referring to Eq. (20), where

$$a^- \sim P - i\omega x\,, \tag{22}$$

which converts to

$$(P - i\omega x)\psi = 0\,. \tag{23}$$

Or

$$\big(- i\frac{d}{dx} - i\omega x\big)\psi = 0\,. \tag{24}$$

And, on cancelling the i's we get

$$\big(\frac{d}{dx} + \omega x\big)\psi = 0\,. \tag{25}$$

To integrate, try the ansatz $\psi(x) = e^{f(x)}$. Then,

$$\frac{d}{dx}e^{f(x)} = -\omega x e^{f(x)}\,, \tag{26}$$

or

$$f'(x) = -\omega x\,. \tag{27}$$

Integrating this, we get

$$f(x) = -\tfrac{1}{2}\omega x^2\,, \tag{28}$$

where we can ignore the constant of integration, as it will only affect ψ as a constant multiplicative factor. Hence,

$$\psi_0(x) = e^{-\frac{1}{2}\omega x^2}\,, \tag{29}$$

which is a gaussian function.

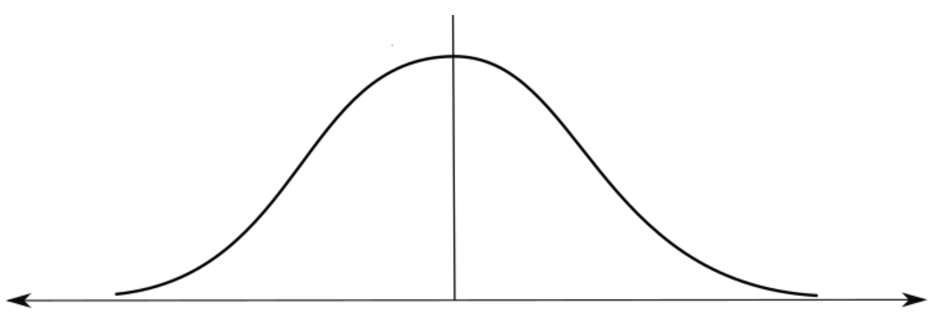

Figure 1. The ground state solution to the Schrödinger equation we found to be a gaussian function.

If we plugged our solution for the ground state function into (19), we would discover that the corresponding energy value would be 1/2, which we already knew from (8).

Now we try to find the first excited state $|\,1\,\rangle$.

$$|\,1\,\rangle = a^+\,|\,0\,\rangle\;, \tag{30}$$

or

$$|\,1\,\rangle = (P + i\omega x)\psi_0(x)\,, \tag{31}$$

or

$$\psi_1(x) = \big(-i\frac{d}{dx} + i\omega x\big)e^{-\frac{1}{2}\omega x^2} = 2i\omega x e^{-\frac{1}{2}\omega x^2}\;. \tag{32}$$

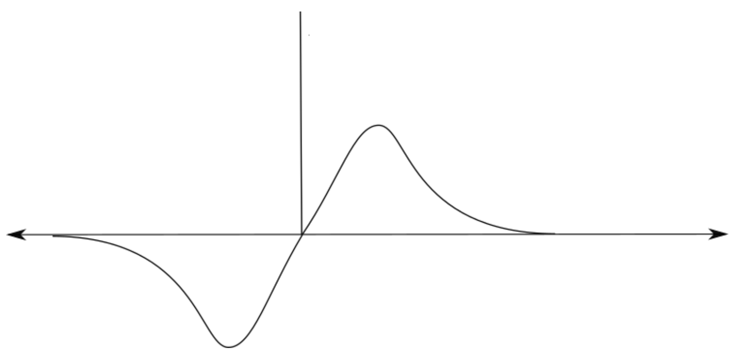

Figure 2. The first excited state solution to the Schrödinger equation.

We saw that the ground state had no nodes, and the first excited state has a single node. And so it goes.

2 Intrinsic spin

Particles with spin 1/2 have the property of angular momentum, as though the particle is spinning about an axis, though most physicists do not take this interpretation literally.

What characterizes a particle's angular momentum? First, that there are three components of it, which we'll associate them with the three directions of cartesian space.

Now,

$$[\,L_x, L_y\,] = i\hbar L_z\;, \tag{33}$$
$$[\,L_y, L_z\,] = i\hbar L_x\;, \tag{34}$$
$$[\,L_z, L_x\,] = i\hbar L_y\;. \tag{35}$$

Now, let's take a look at the Pauli matrices, beginning with σ_z:

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\;. \tag{36}$$

This matrix is rather special among the Pauli matrices, for, besides the identity matrix, it is the only other one that is diagonal. The eigenvalues of this matrix are $+1$ and -1, representing, respectively, up and down. The other Pauli matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \tag{37}$$

So, what are the commutation relations of these matrices? Let's try out σ_z with σ_x:

$$\sigma_z \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_y . \tag{38}$$

And

$$\sigma_x \sigma_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_y . \tag{39}$$

Therefore,

$$[\, \sigma_z, \sigma_x \,] = \sigma_z \sigma_x - \sigma_x \sigma_z = 2i\sigma_y . \tag{40}$$

We can get rid of this factor of 2 by defining a new variable s such that

$$s_i \equiv \tfrac{1}{2}\sigma_i , \qquad \text{for} \qquad i = 1, 2, 3 . \tag{41}$$

Therefore,

$$[\, s_z, s_x \,] = i\hbar s_y , \tag{42}$$

and so on for the other two spin matrices.

Anyway, the upshot of this analysis is that this spin acts analogously to linear momentum, hence, it is some form of angular momentum.

So, what are the eigenvalues of s_z? Since

$$s_z = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \tag{43}$$

its eigenvalues are $+\frac{1}{2}$ and $-\frac{1}{2}$.

How do we relate the angular momentum of a particle (relative to its CM motion) and its intrinsic angular momentum, or spin? We use the symbol $\mathbf{J}$:

$$\mathbf{J} = \mathbf{L} + \mathbf{s} . \tag{44}$$

Now,

$$L^2 = \ell(\ell + 1) . \tag{45}$$

For each ℓ there are $2\ell = 1$ states, but there are additional degeneracies.

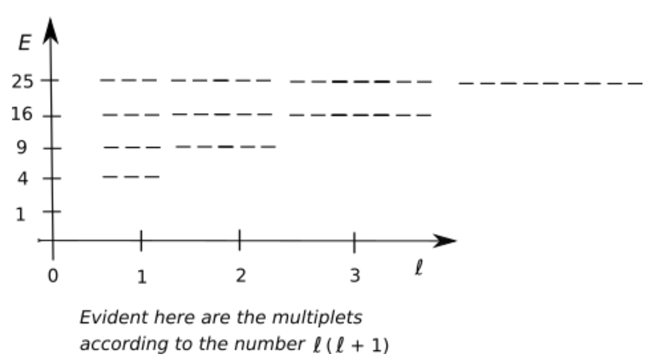

Figure 3. Extra degeneracies in the hydrogen atom.

3 The Pauli Exclusion Principle

In early quantum history there was a conundrum to solve: Observations on the energy states of atomic electrons showed that it is not possible for all the electrons in multi-electron atoms to inhabit the lowest possible energy level. So, the question is, Why? Wolfgang Pauli proposed a technical solution that, although it doesn't explain why this cannot happen, it does give a specific rule to exclude it from happening. The *Pauli Exclusion Principle* can be states as

> It is not possible in a quantum system that any two fermions (spin-$\frac{1}{2}$ particles) of the exact same quantum numbers can share the same restricted space. Any two quantum particles that are in the same space and have the same quantum numbers are said to be in the same 'state'. Bosons are the exceptions to this rule:[1] They may share the same space and quantum numbers.

If this is true, then the two particles that cohabit the lowest energy level of helium, for example, must have different quantum numbers in at least one value. That quantum number of distinction must be their spin numbers: One must be $+\frac{1}{2}$ and the other $-\frac{1}{2}$. These two electron-spin states are respectively referred to as 'up' and 'down'.

To look deeper into the role that particle indistinguishability plays in quantum mechanics, Susskind invented an operator that acts on wave function and produces other wave function. He called this operator the 'swap' operator S, which acts according to

$$S\,|\,x_1 x_2\,\rangle\;, \tag{46}$$

where by $|\,x_1 x_2\,\rangle$ we mean a particle at position 1, and a particle at position 2, reading the arguments from left to right, as usual. Hence,

$$S\,|\,x_1 x_2\,\rangle = |\,x_2 x_1\,\rangle\;. \tag{47}$$

Reasonably, $S^2 = 1$, the identity operation. Now, S is unitary, meaning that it doesn't change probabilities. Now, since $S^2 = 1$, then

$$S = \pm 1\,. \tag{48}$$

We end the lecture here, to pick it up next time at this point.

[1] Bosons are particles of integer spin numbers.